Frames via Unilateral Iterations of Bounded Operators

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Collaborators and Contributors to Dynamical Sampling

- Joint work with Carlos Cabrelli (Universidad de Buenos Aires).
- Main contributors to Dynamical Sampling: Aceska, Aldroubi, Bownik, Cabrelli, Çakmak, Christensen, Hasannasab, Huang, Kim, Kornelson, Krishtal, Molter, Paternostro, Petrosyan, Philipp, Stoeva, Tang.

Background

- $\{f_k\}_{k \in I} \subset H$ is a frame if $\exists$ fixed constants $0 < C_1 \leq C_2$ such that for each $f \in H$,

$$C_1 \|f\|^2 \leq \sum_k |\langle f, f_k \rangle|^2 \leq C_2 \|f\|^2$$

- $U : l^2(I) \to H$ where $U\{c_k\}_k = \sum_k c_k f_k$ is the synthesis operator.
- $U^* : H \to l^2(I)$ where $U^* f = \{\langle f, f_k \rangle\}_k$ is the analysis operator.
- $\Psi = UU^* \in B(H)$ where $\Psi f = \sum_k \langle f, f_k \rangle f_k$ is the frame operator.
- The canonical dual frame is $\{\Psi^{-1} f_k\}_{k \in I}$. For all $f \in H$, $f = \sum_k \langle f, \Psi^{-1} f_k \rangle f_k = \sum_k \langle f, f_k \rangle \Psi^{-1} f_k$
- A Riesz sequence is a system $\{f_k\}_k$ that forms a bounded unconditional basis for $\overline{span}\{f_k\}_k$.

Dynamical Sampling in Infinite Dimensional Hilbert Spaces

- The Dynamical Sampling problem was completely solved for finite dimensional Hilbert spaces in the paper "Dynamical Sampling" by Aldroubi, Cabrelli, Molter, Tang.

- Let H be a separable infinite dimensional Hilbert space.

- In Dynamical Sampling, we seek to recover $f \in H$ using the samples $\{\langle A^n f, g\rangle\}_{0\leq n\leq L(g),\, g\in\mathcal{A}} = \{\langle f, (A^*)^n g\rangle\}_{0\leq n\leq L(g),\, g\in\mathcal{A}}$

- Any $f \in H$ can be stably reconstructed from these samples if $\{(A^*)^n g\}_{0\leq n\leq L(g), g\in\mathcal{A}}$ is a frame for H.

- Given an operator $T \in B(H)$ and a vector $\varphi \in H$, what are conditions for the system $\{T^n\varphi\}_{n\geq 0}$ to be a frame, basis, Bessel, complete, minimal etc.?

Dynamical Sampling in Infinite Dimensional Hilbert Spaces

- If T is normal, then $\{T^n\varphi\}_{n\geq 0}$ can never be a basis. (Aldroubi, Cabrelli, Çakmak, Molter, Petrosyan)

- If T is unitary, then $\{T^n\varphi\}_{n\geq 0}$ can never be a frame. (Aldroubi, Petrosyan)

- If T is compact, then $\{T^n\varphi\}_{n\geq 0}$ can never be a frame. (Christensen, Hasannasab, Rashidi)

- If T is hypercyclic, then $\{T^n\varphi\}_{n\geq 0}$ can never be a frame. (Christensen, Hasannasab)

- If T is self-adjoint, then $\{\frac{T^n\varphi}{\|T^n\varphi\|}\}_{n\geq 0}$ can never be a frame. Conjecture: this holds when T is normal. (Aldroubi, Cabrelli, Çakmak, Molter, Petrosyan)

Dynamical Sampling in Infinite Dimensional Hilbert Spaces

Theorem (Christensen, Hasannasab)

A frame $\{f_n\}_{n\in\mathbb{N}_0}$ admits the form $\{T^n\varphi\}_{n\geq 0}$ with $T \in B(H)$ iff $\{f_n\}_{n\in\mathbb{N}_0}$ is linearly independent and $Ker(U)$ (the kernel of its synthesis operator) is an invariant subspace of the right shift operator, $R \in B(\ell^2(\mathbb{N}_0))$.

$$R(x_0, x_1, x_2 \dots) = (0, x_0, x_1, x_2, \dots).$$

- This result connects Dynamical Sampling to the theory of Hardy Spaces and helps us find necessary and sufficient conditions for $\{T^n\varphi\}_{n\geq 0} \subset H$ to be a frame.

Background

- $H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(z)\overline{z}^n dz = 0 \text{ if } n < 0\ \}$.
- Given $f \in H^2(\mathbb{T})$, $f(z) = \sum_{n \geq 0} c_n z^n$.
- An inner function, θ, is a function in $H^2(\mathbb{T})$ such that $|\theta(z)| = 1$ almost everywhere.

Theorem (Beurling)

Every nontrivial invariant subspace of the shift operator $S \in B(H^2(\mathbb{T}))$, where $Sf(z) = zf(z)$, is of the form $\theta H^2(\mathbb{T})$ for some inner function θ. Conversely, for any inner function θ, $\theta H^2(\mathbb{T})$ is invariant under S.

Frames via Unilateral Iterations of Bounded Operators

Theorem (VB)

Let $\{f_k\}_k$ be a linearly independent and overcomplete frame. Then $\{f_k\}_k = \{T^n \varphi\}_{n\geq 0}$ with $T \in B(H)$ iff $\{R^n c\}_{n\geq 0}$ is a Parseval frame for $Ker(U)$ for some $c \in \ell^2(\mathbb{N}_0)$ whose image under $A : \ell^2(\mathbb{N}_0) \to H^2(\mathbb{T})$, where $A(c_0, c_1, \dots) = \sum_{n\geq 0} c_n z^n$, is an inner function.

- Invariant subspaces of R correspond to invariant subspaces of S via the unitary map A.
- Beurling's theorem implies that $Ker(U)$ corresponds to an invariant subspace of the form $\theta H^2(\mathbb{T})$. $\theta H^2(\mathbb{T})$ is a cyclic invariant subspace of S so that $Ker(U)$ admits the form above.
- In fact, $\{R^n c\}_{n\geq 0}$ is an orthonormal basis frame for $Ker(U)$.

Frames via Unilateral Iterations of Bounded Operators

Definition

A model space, K_θ, is a subspace of $H^2(\mathbb{T})$ of the form $K_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T})$ for some shift-invariant subspace $\theta H^2(\mathbb{T})$.

Definition

The map $S_\theta = P_{K_\theta} \, S|_{K_\theta}$ is the compression of the shift to K_θ where P_{K_θ} is the orthogonal projection onto K_θ.

Definition

A finite Blaschke product is an inner function of the form $\phi(z) = c\prod_{j=1}^{k} \frac{\lambda_j - z}{1-\overline{\lambda_j}z}$ where $\{\lambda_1, \ldots \lambda_k\} \subset \mathbb{D}$ and $c \in \mathbb{T}$.

Frames via Unilateral Iterations of Bounded Operators

Definition

Let H, K be complex separable infinite-dimensional Hilbert Spaces. Given $T \in B(H)$ and $A \in B(K)$ we say the pairs (T, f) and (A, g) are similar and write $(T, f) \cong (A, g)$ if there exists $L \in GL(H, K)$ such that $LTL^{-1} = A$ and $Lf = g$.

Lemma

Assume $(T, f) \cong (A, g)$. Then $\{T^n f\}_{n \geq 0}$ is a frame (overcomplete frame) for H if and only $\{A^n g\}_{n \geq 0}$ is a frame (overcomplete frame) for K.

Lemma

A model space $K_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T})$ is of finite dimension if and only if the inner function θ is a finite Blaschke product.

Lemma

$\{S_\theta^n P_{K_\theta} 1_{\mathbb{T}}\}_{n \geq 0}$ is a overcomplete frame for K_θ.

Frames via Unilateral Iterations of Bounded Operators

Theorem (Christensen, Hasannasab, Philipp)

A system $\{T^n\varphi\}_{n\geq 0} \subset H$, with $T \in B(H)$ is an overcomplete frame if and only if $(T, \varphi) \cong (S_\theta, P_{K_\theta} 1_{\mathbb{T}})$ for some unique inner function, θ, that is not a finite Blaschke product.

- If $\{T^n\varphi\}_{n\geq 0} \subset H$ is an overcomplete frame and $T \in B(H)$, then $Ker(U)$ is nontrivial and right shift invariant. The kernel of the map $V = U\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform, is then an invariant subspace of $S \in B(H^2(\mathbb{T}))$. Thus $Ker(V) = \theta H^2(\mathbb{T})$ for some inner function θ by Beurling.

- As V is surjective, $Ker(V)$ has infinite codimension. Setting $K_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T})$ and $W = V|_{K_\theta}$ we have $W \in GL(K_\theta, H)$, $WP_{K_\theta} 1_{\mathbb{T}} = \varphi$, and $WS_\theta W^{-1} = T$

Frames via Unilateral Iterations of Bounded Operators

- Let $T_1, T_2 \in B(H)$ commute. We seek necessary and sufficient conditions for a system $\{T_1^i T_2^j f_0\}_{i,j\geq 0} \subset H$ to be a frame.
- $H^2(\mathbb{T}^2) = \{f(z,w) \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f(z,w)\overline{z}^m\overline{w}^n d\mu = 0$ if $m < 0$ or $n < 0\}$.
- Given $f \in H^2(\mathbb{T}^2)$, $f(z,w) = \sum_{i,j\geq 0} c_{ij} z^i w^j$.
- A subspace $M \subseteq H^2(\mathbb{T}^2)$ is shift-invariant if it is invariant under both shift operators S_1 and S_2. That is $S_1 M = zM \subset M$ and $S_2 M = wM \subset M$.
- An inner function, θ, is a function in $H^2(\mathbb{T}^2)$ such that $|\theta(z,w)| = 1$ almost everywhere.
- Beurling's characterization of invariant subspaces does not translate fully to $H^2(\mathbb{T}^2)$.

Frames via Unilateral Iterations of Bounded Operators

Theorem (Mandrekar)

A non-trivial shift-invariant subspace $M \subset H^2(\mathbb{T}^2)$ is of the form $\phi H^2(\mathbb{T}^2)$ with $\phi(z,w)$ inner if and only if S_1 and S_2 doubly commute on M.

Lemma

Let $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ be an overcomplete frame for H where $T_1, T_2 \in B(H)$ commute. Let $U : \ell^2(\mathbb{N}_0 \times \mathbb{N}_0) \to H$ be the synthesis operator. Let $R_1, R_2 \in B(\ell^2(\mathbb{N}_0 \times \mathbb{N}_0))$ be the right shift in the first and second components respectively. If R_1, R_2 doubly commute on $Ker(U)$, then the invariant subspace $Ker(V) = Ker(U\mathcal{F}) \subset H^2(\mathbb{T}^2)$ is of the form $\phi H^2(\mathbb{T}^2)$, where $\phi(z,w)$ is an inner function.

Frames via Unilateral Iterations of Bounded Operators

- By Argawal, Clark, and Douglas when $\phi(z, w)$ is an inner function, any invariant subspace $M \subseteq H^2(\mathbb{T}^2)$ satisfies $\phi M \subseteq M$ with equality if and only if ϕ is constant. Also, they show that every invariant subspace $M \subseteq H^2(\mathbb{T}^2)$ with finite codimension has full range.

- By Mandrekar, invariant subspaces of the form $\phi H^2(\mathbb{T}^2)$ do not have full range unless ϕ is constant. Thus, a Beurling type invariant subspace, $\phi H^2(\mathbb{T}^2)$, cannot have finite codimension unless ϕ constant. That is, invariant subspaces of the form $\phi H^2(\mathbb{T}^2)$ with $\phi(z, w)$ inner, satisfy $dim(H^2(\mathbb{T}^2) \ominus \phi H^2(\mathbb{T}^2)) = \infty$ unless $\phi H^2(\mathbb{T}^2) = H^2(\mathbb{T}^2)$.

Frames via Unilateral Iterations of Bounded Operators

Theorem (Cabrelli, VB)

Let $\{T_1^i T_2^j f_0\}_{i,j\geq 0} \subset H$, where $T_1, T_2 \in B(H)$ commute, satisfy the property that the operators R_1, R_2 doubly commute on $Ker(U)$, where U is the synthesis operator for the sequence $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$. Then $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ is an overcomplete frame iff there exists a nonconstant inner function, $\theta(z,w)$, such that $(T_1 T_2, f_0) \cong (S_{\theta_1} S_{\theta_2}, P_{K_\theta} 1_{\mathbb{T}^2})$ where P_{K_θ} is the orthogonal projection onto $K_\theta = H^2(\mathbb{T}^2) \ominus \theta H^2(\mathbb{T}^2)$ and $S_{\theta_1} = P_{K_\theta}\, S_1|_{K_\theta}$ and $S_{\theta_2} = P_{K_\theta}\, S_2|_{K_\theta}$.

- A more general version of this theorem holds without assuming double commuting property.
- Theorem provides characterization of frames obtained by iterations of pairs of commuting bounded operators.

Future Work on Frames via Operator Orbits

- Characterize bounded, commuting operators T_1, T_2 that for some $\varphi \in H$, admit double commuting property on $Ker(U)$.
- Find necessary and sufficient conditions for systems given by iterations of bounded operators that do not necessarily commute to be a frame.
- Determine whether all shift-invariant subspaces of $H^2(\mathbb{T}^2)$ that are not full range admit the form ϕN, for some $N \subseteq H^2(\mathbb{T}^2)$.

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