Rational approximation to analytic functions with polar singular set and finitely many branchpoints

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based on joint work with

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- Runge's proof rests on his "pole shifting technique".
- Today, it is a consequence of the duality between complex measures and continuous functions with compact support.

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  - in electrical engineering, to check stability of microwave circuits.

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- Hence, rational approximation may be viewed as optimal discretization of a logarithmic potential with respect to a Sobolev norm.

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- We set

$$e_n = e_n(f, K) := \inf_{r_n \in \mathcal{R}_n} \|f - r_n\|_{L^\infty(K)}.$$

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- n-th root rates only estimate the geometric decay of the error.
- They make contact with logarithmic potential theory.

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- ω_K is characterized by V^{ω_K} being constant q.e. on K (Frostman theorem).

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- The capacity of a set E is the supremum of C_K over all compact $K \subset E$.

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- When $\psi \equiv 0$ one recovers the usual capacity.

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$$\Delta u = -\delta_z \quad \text{in } \Omega.$$
- Example: if $\mathbb{D}$ is the unit disk, then
$$G_{\mathbb{D}}(z,t) = \log\left|\frac{1-z\bar{t}}{z-t}\right|.$$

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$$\left(= \|\nabla V_\Omega^\mu\|^2_{L^2(\Omega)} \text{ in smooth cases }\right)$$

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- Green capacities and Green equilibrium distributions are conformally invariant. This allows to speak of the Green capacity of a closed set, possibly containing ∞, in an open set of the Riemann sphere.

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- It is obtained by interpolating the function. There are functions for which this bound is sharp (Tikhomirov).

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- For this they used interpolation again.

Padé interpolants and N.H. orthogonal polynomials

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- Note that orthogonality is non Hermitian.
- To assess the asymptotic behavior of q_n, it was realized that E should have special properties. of the normalized counting measures of the $\xi_j^{(n)}$:

$$\frac{1}{2n} \sum_{\ell=1}^{2n} \delta_{\xi_\ell^{(n)}} \xrightarrow{w*} \nu.$$

Symmetric contours

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- Dwelling on his work, Gonchar and Rakhmanov generalized the result to the multipoint case as follows.

The Gonchar-Rakhmanov theorem

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Theorem [Gonchar-Rachmanov,87] If f is (essentially) a Cauchy integral on a weighted S-contour E in the field $-V^{\nu}$, with q.e. continuous nonzero density on E, and if the interpolation points $\xi_1^{(n)}, \cdots, \xi_{2n}^{(n)}$ are picked with asymptotic density ν on K:

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and the normalized counting measure of their poles converges towards $\omega_{E,-U^{\nu}}$.

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- Taking moduli and $2n$-th root, the surviving term is

$$\left|\frac{w_{2n}}{q_n^2}(z)\right|^{1/2n}=\exp\left\{\frac{\log|w_{2n}(z)|}{2n}-\frac{\log|q_n(z)|}{n}\right\}$$

$$=\exp\left\{-V^{\nu_n}+V^{\mu_n}\right\}\to\exp\left\{V_G^{\omega^G_{K,E^c}}(z)\right\}.=\frac{1}{C(K,E)}\ \text{on}\ K.$$

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- Is this also the behaviour of best approximants?

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p_n/q_n interpolates f at $1/\bar{\xi}_j$ with order 2 for each zero ξ_j of q_n, and also at 0 [Levin, 76], [Della-Dora, 72]. So, it is a particular case of multipoint Padé interpolant (with unknown interpolation points).

Asymptotics of H^2 approximants

Theorem (H. Stahl, M. Yattselev, L.B., 2013)

Let f be analytic except for finitely many branchpoint off the closed unit disk, and p_n/q_n a best rational approximant of degree n in H^2. The counting measure of the poles of p_n/q_n converges weak- when $n \to \infty$ to the equilibrium measure of the set K_G of minimal Green capacity in $\overline{\mathbb{C}} \setminus \mathbb{D}$ outside of which f is single-valued.*

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The result is not a direct consequence of the Gonchar Rakhmanov theorem, for one needs to establish that there exists a S-contour in the field $-V^\nu$, where ν is the asymptotic distribution of the reciprocal of the poles of p_{n-1}/q_n (the interpolation points).

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- In other words, we approximate f on ∂K by the trace of a meromorphic function with at most n poles in K^c. This makes conformal invariance obvious.
- By the Cauchy formula

$$f(z) - r_n(z) = \frac{1}{2i\pi} \int_{\partial K} \frac{(f - r_n - g)(t)}{t - z}\, dt \quad \text{for } z \in \overset{\circ}{K},$$

which implies easily that

$$\limsup e_{n_k}^{1/n_k} = \limsup em_{n_k}^{1/n_k}, \quad \liminf e_{n_k}^{1/n_k} = \liminf em_{n_k}^{1/n_k}$$

along any subsequence n_k.

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- By outer regularity of capacity, one may further assume that ∂E is a smooth Jordan curve Γ.

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- By conformal mapping assume $K = \overline{\mathbb{C}} \setminus \mathbb{D}$ with $\mathbb{D}$ the unit disk, and $\Omega = \overline{\mathbb{C}} \setminus E$, with E compact lying interior to the unit circle $\mathbb{T}$.
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where $\mathbf{P}_-$ is the projection $L^2(\mathbb{T}) \to H_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ in the orthogonal decomposition:

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- By the above, Fubini's theorem, and the residue formula, we get for $v \in H^2(\mathbb{D})$:

$$A_f(v)(z) = \frac{1}{2i\pi}\int_\Gamma \frac{v(\xi)f(\xi)}{(z-\xi)}\,d\zeta, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

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- B_3, B_2 are bounded, and for the singular values of B_1, B_4 we have [Zakharyuta-Skiba, 1976][Fischer-Micchelli, 1980]:

$$\lim_{k\to\infty} s_k^{1/k}(B_1) = \lim_{k\to\infty} s_k^{1/k}(B_4) = \exp\left(-\frac{1}{C(\overline{\mathbb{C}} \setminus \mathbb{D}, \Gamma)}\right).$$

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- Spectral theoretic interpretation of "2".

Parfenov's proof cont'd

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- Applying now the Horn-Weyl inequalities:

$$\Pi_{k=0}^{n} s_k(AB) \leq \Pi_{k=0}^{n} s_k(A)\, \Pi_{k=0}^{n} s_k(B), \qquad n \in \mathbb{N}$$

valid for any pair of bounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces,

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- from which Parfenov's theorem follows easily upon taking $1/n^2$-roots.

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- Dwelling on Horn-Weyl inequalities for singular values of the Hankel operator with symol f, we prove:

$$\limsup_{n\to\infty} e_n^{1/n} > \exp\left\{\frac{-2}{C(K,\Omega)}\right\} \Longrightarrow \liminf_{n\to\infty} e_n^{1/n} < \exp\left\{\frac{-2}{C(K,\Omega)}\right\}.$$

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- In a second step, one shows that along any subsequence $\liminf e_n^{1/n} \geq \exp\left\{\frac{-2}{C(K,\Omega^*)}\right\}$ and that this speed of convergence is attained only if the asymptotic density of the poles is $\omega^G_{(K,\Omega^*)}$

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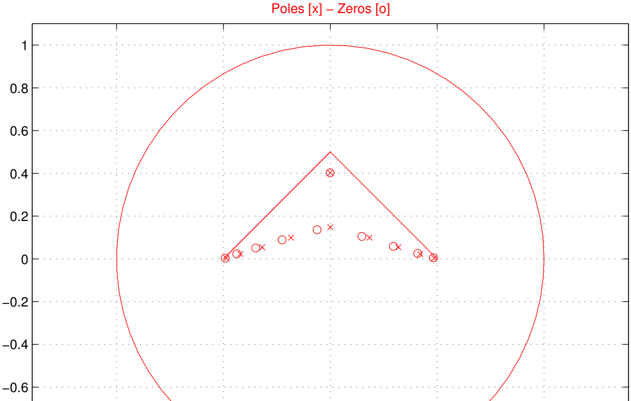
- One dificulty is that L is only finely continuous, but neither subharmonic nor superharmonic.
- In a final step we connect poles in rational approximation with poles in meromorphic approximation. The result on the poles holds in fact for any sequence of approximant with optimal n-th root rate.

Some experiments



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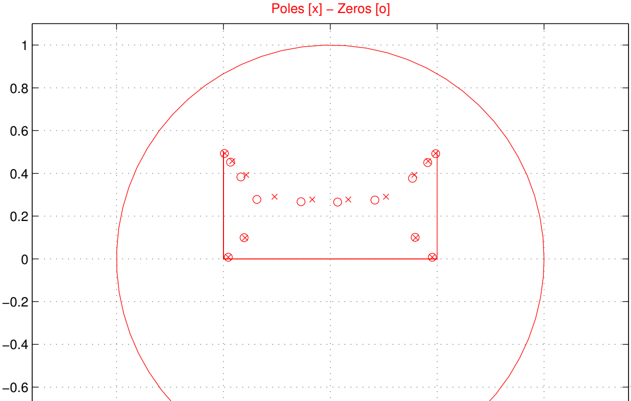

A sad note

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In memoriam Herbert Stahl, August 3, 1942–April 22, 2013.

And most importantly

Thank you!