Asymptotics of Rational Solutions to Painlevé III.

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Example: Consider the differential equation

$$\frac{d^2 u}{dx^2} + \left(\frac{du}{dx}\right)^2 = 0.$$

This equation has a general solution of the form

$$u(x) = \log\left(Ax + B\right)$$

where A, B are arbitrary constants which can be determined via initial conditions.

There are 50 second order equations of the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = R\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}\right),$$

where R is rational in u, u' and analytic in x, whose **only movable singularities are poles.**

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 6u^2(x) + x, \tag{P$_1$}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 2u^3(x) + xu(x) + \alpha, \tag{P$_2$}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{1}{u(x)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{u'(x)}{x} + \frac{\alpha u^2(x) + \beta}{x} + \gamma u^3(x) + \frac{\delta}{u(x)}, \tag{P$_3$}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{1}{2u(x)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{3}{2}u^3(x) + 4xu^2(x) + 2(x^2 - \alpha)u(x) + \frac{\beta}{u(x)}, \tag{P$_4$}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{3u(x) - 1}{2u(x)(u(x) - 1)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{(u(x) - 1)^2}{x^2}\left(\alpha u(x) + \frac{\beta}{u(x)}\right) + \frac{\gamma u(x)}{x} + \frac{\delta u(x)(u(x) + 1)}{u(x) - 1} \tag{P$_5$}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{1}{2}\left(\frac{1}{u(x)} + \frac{1}{u(x) - 1} + \frac{1}{u(x) - x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \left(\frac{1}{x} + \frac{1}{x - 1} + \frac{1}{u(x) - x}\right)\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{u(x)(u(x) - 1)(u(x) - x)}{x^2(x - 1)^2}\left(\alpha + \frac{\beta x}{u^2(x)} + \frac{\gamma(x - 1)}{(u(x) - 1)^2} + \frac{\delta x(x - 1)}{(u(x) - x)^2}\right) \tag{P$_6$}$$

In this talk, we'll focus our attention on P_3,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{1}{u(x)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{u'(x)}{x} + \frac{\alpha u^2(x) + \beta}{x} + \gamma u^3(x) + \frac{\delta}{u(x)}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

This is the simplest of the six Painlevé equations which has a fixed singular point at $x = 0$.

Remark

- $\mathrm{P}_3(D_6)$: $\gamma\delta \neq 0$; this is the generic choice,
- $\mathrm{P}_3(D_7)$: $\gamma = 0$ or $\delta = 0$ but not both,
- $\mathrm{P}_3(D_8)$: $\gamma = \delta = 0$. Here, we can take $\alpha = \beta = 4$.

In this talk, we'll focus our attention on P_3,

$$\frac{d^2u}{dx^2} = \frac{1}{u(x)}\left(\frac{du}{dx}\right)^2 - \frac{u'(x)}{x} + \frac{\alpha u^2(x) + \beta}{x} + \gamma u^3(x) + \frac{\delta}{u(x)}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

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Remark

Up to a change of variables, $\mathrm{P}_3(D_6)$ can be written as

$$\frac{d^2u}{dx^2} = \frac{1}{u(x)}\left(\frac{du}{dx}\right)^2 - \frac{u'(x)}{x} + \frac{4\Theta_0 u^2(x) + 4(1-\Theta_\infty)}{x} + 4u^3(x) - \frac{4}{u(x)}.$$

Rational Solutions of P_3

- Generic solutions of P_3 are highly transcendental.
- However, P_2–P_6 all possess special solutions written in terms of elementary and/or classical special functions.

Proposition. Let

$$\Theta_0 = n + m, \quad \Theta_\infty = m - n + 1.$$

If P_3 has a rational solution, then, $n \in \mathbb{Z}$ or $m \in \mathbb{Z}$.

Rational Solutions of P_3

Proposition. Let

$$\Theta_0 = n + m, \quad \Theta_\infty = m - n + 1.$$

If P_3 has a rational solution, then, $n \in \mathbb{Z}$ or $m \in \mathbb{Z}$.

Proof. Suppose $u(x)$ is rational. Then, as $x \to \infty$, we may write

$$u(x) = ax^p + O(x^{p-1}).$$

Plugging this into P_3 gives a "dominant balance" when $p = 0$, and $a^4 = 1$. Or,

$$u(x) = a + bx^{-1} + O(x^{-2}) \quad \text{as} \quad x \to \infty.$$

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Plugging this back in gives $b = a^2(\Theta_\infty - 1)/4 - \Theta_0/4$. A similar computation near any pole $x_0 \neq 0$ yields that

$$u(x) = \frac{\pm 1}{2(x - x_0)} + O(1) \quad \text{as} \quad x \to x_0.$$

Letting $k \in \mathbb{Z}$ be the difference between number of poles with residues $1/2$ and $-1/2$,

$$\frac{k}{2} + a^2 \frac{1}{4}(\Theta_\infty - 1) - \frac{1}{4}\Theta_0 = 0.$$

When $a^2 = 1$, this gives $k = n$; when $a^2 = -1$ this gives $k = m$.

Some more facts (Recall that $\Theta_0 = n + m,\ \Theta_\infty = m - n + 1$)

- When $n \in \mathbb{Z}$ or $m \in \mathbb{Z}$ (but not both), P_3 has two rational solutions.
  - When $n = 0, m \notin \mathbb{Z}$, the rational solutions are $u(x) = \pm 1$.
  - When $m = 0, n \notin \mathbb{Z}$, the rational solutions are $u(x) = \pm \mathbf{i}$.
- When $m = n = 0$, then both $u(x) = \pm 1,\ u(x) = \pm \mathbf{i}$ are solutions.

Bäcklund Transformations

To move between solutions, $(\Theta_0 = n + m, \;\; \Theta_\infty = m - n + 1)$

- **Inversion:**

$$I : (u(x), \Theta_0, \Theta_\infty) \mapsto (1/u(x), \Theta_\infty - 1, \Theta_0 + 1),$$
$$I : (u(x), m, n) \mapsto (1/u(x), m, -n).$$

- **Rotation:**

$$R : (u(x), \Theta_0, \Theta_\infty) \mapsto (-\mathrm{i}u(-\mathrm{i}x), \Theta_0, 2 - \Theta_\infty),$$
$$R : (u(x), m, n) \mapsto (-\mathrm{i}u(-\mathrm{i}x), n, m).$$

- **1-Step:**

$$G : (u(x), \Theta_0, \Theta_\infty) \mapsto (\hat{u}(x), \Theta_0 + 1, \Theta_\infty - 1),$$
$$G : (u(x), m, n) \mapsto (\hat{u}(x), m, n + 1)$$

Bäcklund Transformations

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- **1-Step:**

$$G : (u(x), \Theta_0, \Theta_\infty) \mapsto (\hat{u}(x), \Theta_0 + 1, \Theta_\infty - 1),$$
$$G : (u(x), m, n) \mapsto (\hat{u}(x), m, n + 1)$$

Here[1],

$$\hat{u}(x) := \frac{xu'(x) + 2xu^2(x) + 2x - 2(1 - \Theta_\infty)u(x) - u(x)}{u(x)(xu'(x) + 2xu^2(x) + 2x + 2\Theta_0 u(x) + u(x))}.$$

[1] V. I. Gromak, The solutions of Painlevé's third equation, *Differencial'nye Uravnenija* (1973)

Proposition.[2,3] The result of applying the Bäcklund transformation G to $u(x) \equiv 1$ n times is

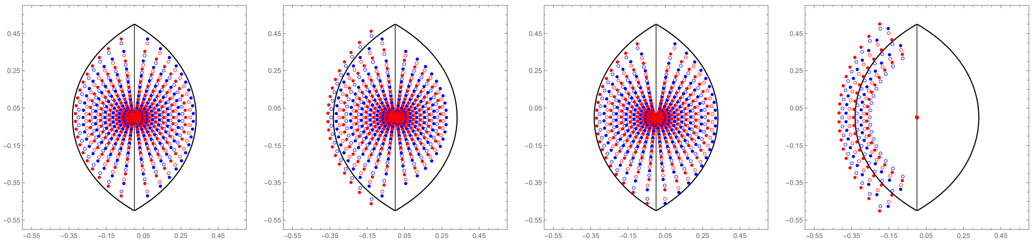
$$u_n(x;m) := \frac{s_n(x;m-1)s_{n-1}(x;m)}{s_n(x;m)s_{n-1}(x;m-1)}, \quad n \in \mathbb{N},$$

where $\{s_n(x;m)\}_{n=0}^{\infty}$ are the *Umemura Polynomials* defined by $s_{-1}(x;m) = s_0(x;m) \equiv 1$ and

$$s_{n+1}(x;m) = \frac{(4x+2m+1)s_n^2(x;m) - s_n(x;m)s_n'(x;m) - x(s_n(x;m)s_n''(x;m) - (s_n'(x;m)^2))}{2s_{n-1}(x;m)}.$$

[2] H. Umemura, Painlevé equations in the past 100 years, Am. Math. Soc. Transl. (2001)

[3] P. A. Clarkson, The third Painlevé equation and associated special polynomials, J. Phys. A: Math. Gen.(2003)



(a) $n = 20,\ m = 0$ (b) $n = 20,\ m = 1$ (c) $n = 20,\ m = \mathrm{i}/2$ (d) $n = 20,\ m = 5/2$

Figure: Zeros (blue)/poles (red) of $u_n(ny; m)$ where $y = x/n$. Courtesy of P. Miller.[4]

[4] Bothner, T.J., Miller, P.D., Sheng, Y.: Rational solutions of the Painlevé-III equation. Stud. Appl. Math. (2018)

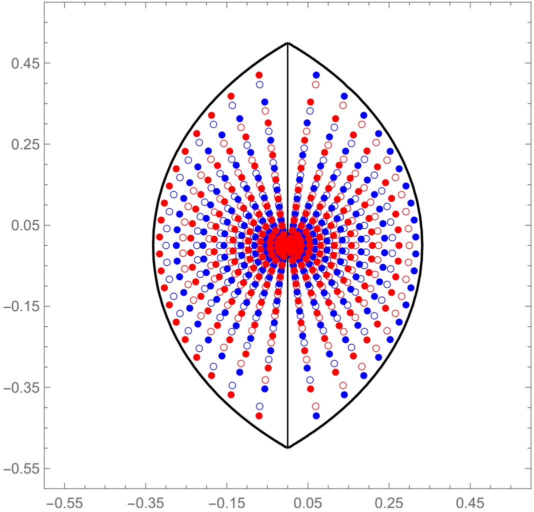


Figure: $n = 20, m = 0$. Zeros (blue)/poles (red) of $u_n(ny; m)$ when $n = 20, m = 0$ where $y = x/n$. Filled blue circles are zeros of $s_n(x; m-1)$, unfilled are zeros of $s_{n-1}(x; m)$. Filled red circles are zeros of $s_n(x; m)$, unfilled are zeros of $s_{n-1}(x; m-1)$.

$u_n(x; m)$ **near** $x = 0$

$u_n(0; m)$

Theorem.[5] Let $\phi_1(\mu) = \mu$ and define

$$\phi_{2k}(\mu) = \phi_{2k-1}(\mu) \prod_{j=1}^{k} (\mu^2 - (2j-1)^2), \qquad \phi_{2k+1}(\mu) = \phi_{2k}(\mu) \cdot \mu \cdot \prod_{j=1}^{k} (\mu^2 - (2j)^2)$$

Then,

$$u_n(0; m) = \frac{\phi_n(m - 1/2)\phi_{n-1}(m + 1/2)}{\phi_n(m + 1/2)\phi_{n-1}(m - 1/2)}.$$

[5] P. A. Clarkson, C.-K. Law, C.-H. Lin, An constructive proof for the Umemura polynomials for the third Painlevé equation, arxiv:1609.00495, 2018

$u_n(0; m)$

$$u_n(0; m) = \frac{\phi_n(m - 1/2)\phi_{n-1}(m + 1/2)}{\phi_n(m + 1/2)\phi_{n-1}(m - 1/2)}.$$

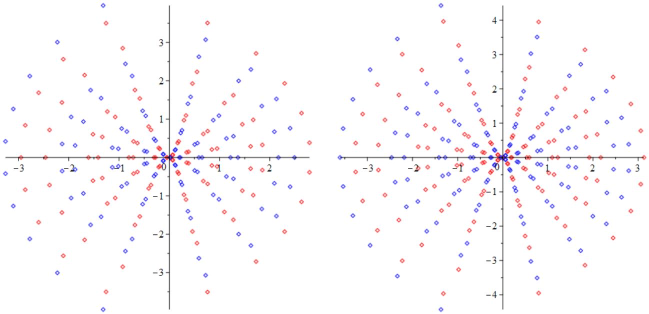
where

$$\begin{aligned}
\phi_1(\mu) &= \mu, \\
\phi_2(\mu) &= \mu(\mu^2 - 1), \\
\phi_3(\mu) &= \mu^2(\mu^2 - 1)(\mu^2 - 4), \\
\phi_4(\mu) &= \mu^2(\mu^2 - 1)(\mu^2 - 4)(\mu^2 - 9), \\
\phi_5(\mu) &= \mu^3(\mu^2 - 1)(\mu^2 - 4)(\mu^2 - 9)(\mu^2 - 16) \\
&\vdots
\end{aligned}$$

Scaling Analysis

To "zoom in" on $x = 0$, we write $x = z/n$. The differential equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \frac{1}{u(z)}\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{4u^2+4}{z} + O(n^{-1})$$



(a) $n = 10,\ m = 0.25$ (b) $n = 11,\ m = 0.25$

Scaling Analysis

To "zoom in" on $x = 0$, we write $x = z/n$. The differential equation becomes

$$\frac{d^2u}{dz^2} = \frac{1}{u(z)}\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{4u^2+4}{z} + O(n^{-1})$$

Conjecture. Fix $m \in \mathbb{C} \setminus (m + \frac{1}{2})$. There exists a solution of $\mathrm{P}_3(D_8)$, $w(z; m)$, so that

$$\lim_{k\to\infty} u_{2k}\left(\frac{z}{2k}; m\right) = w(z; m), \quad \lim_{k\to\infty} u_{2k+1}\left(\frac{z}{2k+1}; m\right) = -w^{-1}(z; m).$$

Limits of Frobenius' Solutions

Theorem. Fix $m \in \mathbb{C}$ and let $\{v_n(z;m)\}_{n=1}^{\infty}$ be a sequence of solutions of $\mathrm{P}_3\,(D_6)$ that are analytic at the origin $z = 0$ and suppose that

$$\lim_{n\to\infty} v_n(0;m) = \xi_{\infty,0} = \xi_{\infty,0}(m) \neq 0.$$

Then, there exists $\rho > 0$ so that for all n sufficiently large $v_n(z;m)$ is analytic for $|z| < \rho$ and such that $v_n(z;m) \to v_\infty(z;m)$ as $n \to \infty$ uniformly for $|z| < \rho$, where $w(z) = v_\infty(z;m)$ solves P_3 $(D8)$ equation, is analytic at the origin, and $v_\infty(z;m) = \xi_{\infty,0}$.

Application to P_3

Let $v(x) = v_n(x; m) := u_n(x/n; m)$. Then,

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{1}{v}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\alpha_n v^2(x)}{x} + \frac{\beta_n}{x} + \gamma_n v^3(x) + \frac{\delta_n}{v(x)}, \qquad (1)$$

where

$$\alpha_n := 4 + \frac{4m}{n}, \quad \beta_n := 4 - \frac{4m}{n}, \quad \gamma_n := \frac{4}{n^2}, \quad \delta_n := -\frac{4}{n^2}.$$

Initial Conditions

Recall that

$$u_n(0;m) = \frac{\phi_n(m-1/2)\phi_{n-1}(m+1/2)}{\phi_n(m+1/2)\phi_{n-1}(m-1/2)},$$

where $\phi_1(\mu) = \mu$ and define

$$\phi_{2k}(\mu) = \phi_{2k-1}(\mu)\prod_{j=1}^{k}(\mu^2-(2j-1)^2), \qquad \phi_{2k+1}(\mu) = \phi_{2k}(\mu)\cdot\mu\cdot\prod_{j=1}^{k}(\mu^2-(2j)^2).$$

It follows that

$$\lim_{k\to\infty} u_{2k}(0;m) = \tan\left(\frac{(2m+1)\pi}{4}\right), \qquad \lim_{k\to\infty} u_{2k+1}(0;m) = -\cot\left(\frac{(2m+1)\pi}{4}\right)$$

Initial Conditions

$$\lim_{k\to\infty} u_{2k}(0;m) = \tan\left(\frac{(2m+1)\pi}{4}\right), \qquad \lim_{k\to\infty} u_{2k+1}(0;m) = -\cot\left(\frac{(2m+1)\pi}{4}\right)$$

Corollary. Let $m \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$ and denote by $w(z;m)$ the unique solution of P_3 (D_8) with $w(0;m) = \tan\big((2m+1)\pi/4\big)$. Then

$$\lim_{k\to\infty} u_{2k}\left(\frac{z}{2k};m\right) = w(z;m),$$

$$\lim_{k\to\infty} u_{2k+1}\left(\frac{z}{2k+1};m\right) = -\frac{1}{w(z;m)}.$$

Corollary. Let $m \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$ and denote by $w(z; m)$ the unique solution of P_3 (D_8) with $w(0; m) = \tan\big((2m+1)\pi/4\big)$. Then

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$$\lim_{k\to\infty} u_{2k+1}\left(\frac{z}{2k+1}; m\right) = -\frac{1}{w(z; m)}.$$

Remark

If $w(z)$ solves $\mathrm{P}_3\,(D_8)$, then so does $-1/w(z)$.


Go deeper...

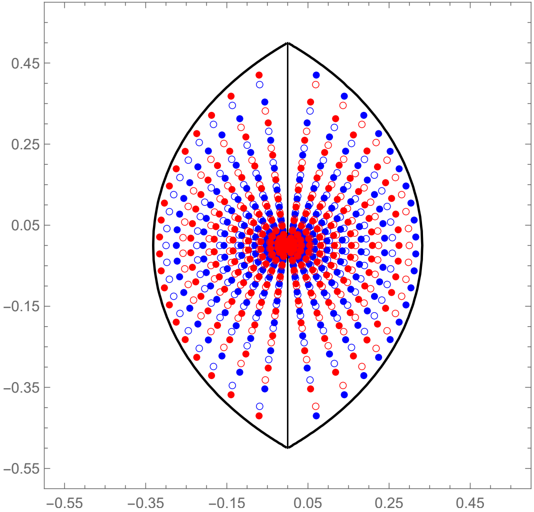


Figure: $n = 20, m = 0$. Zeros (blue)/poles (red) of $u_n(ny; m)$ when $n = 20, m = 0$ where $y = x/n$. Filled blue circles are zeros of $s_n(x; m-1)$, unfilled are zeros of $s_{n-1}(x; m)$. Filled red circles are zeros of $s_n(x; m)$, unfilled are zeros of $s_{n-1}(x; m-1)$.

Different Characterization.

Given $m \in \mathbb{C}$ and $n \in \mathbb{Z}$ as well as $x \in \mathbb{C} \setminus \{0\}$ with $-\pi < \arg(x) < \pi$, we seek a 2×2 matrix $\boldsymbol{Y}(\lambda) = \boldsymbol{Y}_n(\lambda; x, m)$ satisfying

1. $\boldsymbol{Y}(\lambda)$ is analytic in $\mathbb{C} \setminus L$.
2. $\boldsymbol{Y}(\lambda)$ has continuous boundary values on $L \setminus \{0\}$ that satisfy

$$\boldsymbol{Y}_+(\lambda) = \boldsymbol{Y}_-(\lambda) \boldsymbol{J}_{\boldsymbol{Y}}(\lambda). \tag{2}$$

3. $\boldsymbol{Y}(\lambda) \to \mathbb{I}$ as $\lambda \to \infty$ and $\boldsymbol{Y}(\lambda)\lambda_\downarrow^{-(\Theta_0+\Theta_\infty)\sigma_3/2} = \boldsymbol{Y}(\lambda)\lambda_\downarrow^{-(m+\frac{1}{2})\sigma_3}$, where $\Theta_\infty = m - n + 1$, $\Theta_0 = n + m$, has a well-defined limit as $\lambda \to 0$.

Jump Matrix

$$\boldsymbol{J_Y}(\lambda) \in \left\{ \begin{pmatrix} 1 & -\dfrac{\sqrt{2\pi}\lambda_\downarrow^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^n e^{ix(\lambda-\lambda^{-1})} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \dfrac{\sqrt{2\pi}\lambda_\downarrow^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^n e^{ix(\lambda-\lambda^{-1})} \\ 0 & 1 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 0 \\ \dfrac{\sqrt{2\pi}(\lambda_\downarrow^{(m+1)/2})_+(\lambda_\downarrow^{(m+1)/2})_-}{\Gamma(\frac{1}{2}+m)}\lambda^{-n}e^{-ix(\lambda-\lambda^{-1})} & 1 \end{pmatrix},$$

$$\left. \begin{pmatrix} -e^{2\pi i m} & 0 \\ \dfrac{\sqrt{2\pi}(\lambda_\downarrow^{(m+1)/2})_+(\lambda_\downarrow^{(m+1)/2})_-}{\Gamma(\frac{1}{2}+m)}\lambda^{-n}e^{-ix(\lambda-\lambda^{-1})} & -e^{-2\pi i m} \end{pmatrix} \right\}$$

Connection with P_3

If we write

$$\boldsymbol{Y}(\lambda) = \mathbb{I} + \lambda^{-1}\boldsymbol{Y}_1^{\infty} + O(\lambda^{-2}), \qquad \boldsymbol{Y}(\lambda)\lambda_{\downarrow}^{-(\Theta_0+\Theta_\infty)} = \boldsymbol{Y}_0^0 + \lambda\boldsymbol{Y}_1^0 + O(\lambda^2)$$

Then,[5]

$$u_n(x;m) = \frac{-\mathrm{i}\,Y_{1,12}^{\infty}(x)}{Y_{0,11}^0(x)\,Y_{0,12}^0(x)},$$

where $u_n(x;m)$ is the rational solution of P_3 obtained by applying Gromak's Bäcklund transformation to $u(x) \equiv 1$ n times.

[5] Bothner, T.J., Miller, P.D., Sheng, Y.: Rational solutions of the Painlevé-III equation. Stud. Appl. Math. (2018)

When $x \to 0 \ldots$

- One can check that

$$(\boldsymbol{C}^{+}_{0\infty})^{-1}(\boldsymbol{S}^{\infty}_{\uparrow})^{-1}\boldsymbol{C}^{-}_{0\infty}\boldsymbol{S}^{0}_{\uparrow} = \mathbb{I},$$
$$\boldsymbol{C}^{+}_{0\infty}\boldsymbol{S}^{0}_{\downarrow,n}(\boldsymbol{C}^{-}_{0\infty})^{-1}(\boldsymbol{S}^{\infty}_{\downarrow,n})^{-1} = \mathbb{I}.$$

- The Riemann-Hilbert Problem we have is not adequate for studying $x \to 0$.

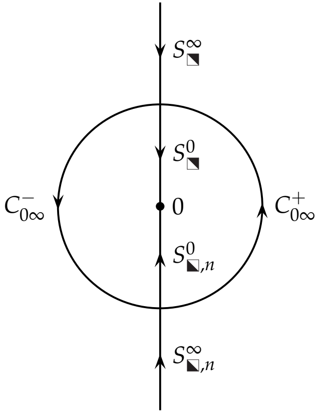
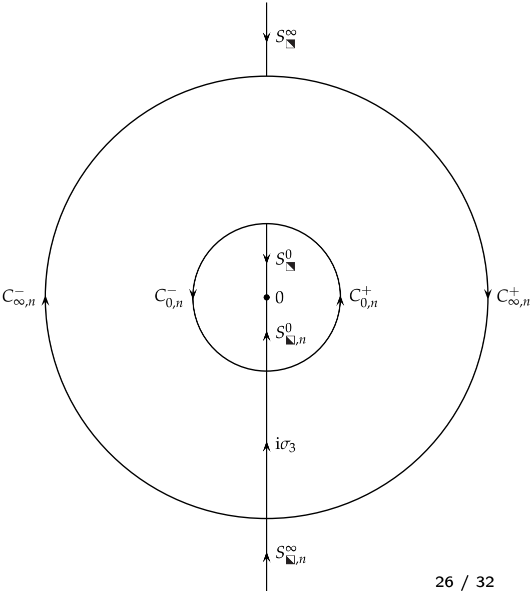


Figure: Contour L when $x \to 0$.

- This step is sponsored by the identities

$$
\begin{aligned}
(\boldsymbol{S}^{\infty}_{\uparrow})^{-1}(\boldsymbol{S}^{\infty}_{\downarrow,n})^{-1} &= (\boldsymbol{C}^{+}_{\infty,n})^{-1}(-\mathrm{i}\sigma_3)\boldsymbol{C}^{+}_{\infty,n},\\
(\boldsymbol{S}^{0}_{\uparrow})^{-1}(\boldsymbol{S}^{0}_{\downarrow,n})^{-1} &= (\boldsymbol{C}^{+}_{0,n})^{-1}(-\mathrm{i}\sigma_3)\boldsymbol{C}^{+}_{0,n},\\
(\boldsymbol{S}^{\infty}_{\downarrow,n})^{-1}(\boldsymbol{S}^{\infty}_{\uparrow})^{-1} &= (\boldsymbol{C}^{-}_{\infty,n})^{-1}(-\mathrm{i}\sigma_3)\boldsymbol{C}^{-}_{\infty,n},\\
(\boldsymbol{S}^{0}_{\downarrow,n})^{-1}(\boldsymbol{S}^{0}_{\uparrow})^{-1} &= (\boldsymbol{C}^{-}_{0,n})^{-1}(-\mathrm{i}\sigma_3)\boldsymbol{C}^{-}_{0,n},\\
\boldsymbol{C}^{+}_{0\infty} &= (\boldsymbol{C}^{+}_{\infty,n})^{-1}\boldsymbol{C}^{+}_{0,n}\\
\boldsymbol{C}^{-}_{0\infty} &= (\boldsymbol{C}^{-}_{\infty,n})^{-1}\boldsymbol{C}^{-}_{0,n},
\end{aligned}
$$



We momentarily ignore the jumps on the inner circles, extend the rays towards the origin and seek a matrix satisfying the described jumps and

$$\mathbf{\Phi}_n^{(\infty)}(\lambda, x) = \begin{cases} \left(\mathbb{I} + \frac{\boldsymbol{A}_n(x)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right) \mathrm{e}^{\mathrm{i}x\lambda\sigma_3/2} \lambda_{\downarrow}^{(n-m-1)\sigma_3/2}, & \lambda \to \infty, \\ \left(\boldsymbol{B}_n(x) + \mathcal{O}(\lambda) \right) \lambda_{\downarrow}^{Q_\infty \sigma_3}, & \lambda \to 0. \end{cases} \tag{3}$$

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It turns out that $\boldsymbol{\Phi}_n^{(\infty)}(\lambda, x)$ satisfies

$$\frac{\partial \boldsymbol{\Phi}_n^{(\infty)}}{\partial \lambda}(\lambda, x) = \left(\frac{\mathrm{i}x}{2}\sigma_3 + \frac{Q_\infty}{\lambda} \boldsymbol{B}_n(x) \sigma_3 \boldsymbol{B}_n(x)^{-1}\right) \boldsymbol{\Phi}_n^{(\infty)}(\lambda, x)$$

which is equivalent to Whittaker's differential equation.

Solving the inner and outer RHP yields a problem on a circle for $\boldsymbol{Q}_n(\lambda, x)$, where

- $\boldsymbol{Q}_n(\lambda, x) \to \mathbb{I}$ as $\lambda \to \infty$ and
- $\boldsymbol{Q}_{n,+}(\lambda, x) = \boldsymbol{Q}_{n,-}(\lambda, x) \boldsymbol{V}_{\boldsymbol{Q}_n}(\lambda, x)$

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- $\boldsymbol{Q}_{n,+}(\lambda, x) = \boldsymbol{Q}_{n,-}(\lambda, x)\boldsymbol{V}_{\boldsymbol{Q}_n}(\lambda, x)$

Proposition. For all $n \in \mathbb{N}$, the Riemann-Hilbert problem for $\boldsymbol{Q}_n(\lambda, x)$ admits an explicit asymptotic expansion for $x \to 0$, which yields

$$u_n(0; m) = \frac{n - m - \frac{1}{2}}{(m - n + \frac{3}{2})(m + n + \frac{1}{2})} 2^{2n} \frac{\Gamma(-\frac{1}{2} - m - n)}{\Gamma(-\frac{3}{2} - m + n)} \frac{\Gamma(-\frac{1}{4} - \frac{m}{2} + \frac{n}{2})^2}{\Gamma(-\frac{1}{4} - \frac{m}{2} - \frac{n}{2})^2}.$$

Proposition. For all $n \in \mathbb{N}$, the Riemann-Hilbert problem for $\boldsymbol{Q}_n(\lambda, x)$ admits an explicit asymptotic expansion for $x \to 0$, which yields

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Contrast this with the Clarkson-Law-Lin formula

$$u_n(0; m) = \frac{\phi_n(m - 1/2)\phi_{n-1}(m + 1/2)}{\phi_n(m + 1/2)\phi_{n-1}(m - 1/2)}.$$

$$\phi_{2k}(\mu) = \phi_{2k-1}(\mu) \prod_{j=1}^{k} (\mu^2 - (2j - 1)^2), \quad \phi_{2k+1}(\mu) = \phi_{2k}(\mu) \cdot \mu \cdot \prod_{j=1}^{k} (\mu^2 - (2j)^2)$$

Limiting Riemann-Hilbert Problem.

- The jump matrix $\boldsymbol{V}_{\boldsymbol{Q}_n}(\lambda, x)$ possesses a limit as $n \to \infty$ with $x = z/n$.
- The resulting Riemann-Hilbert problem for $\boldsymbol{Q}(\lambda, x)$ can be shown to have a solution which is meromorphic in x.

Lax Pair.

In fact, elementary transformations take $\boldsymbol{Q} \mapsto \boldsymbol{\Omega}$ which solves

$$\frac{\partial \boldsymbol{\Omega}}{\partial \lambda}(\lambda, z) = \boldsymbol{\Lambda}(\lambda, z)\boldsymbol{\Omega}(\lambda, z) \quad \text{and} \quad \frac{\partial \boldsymbol{\Omega}}{\partial z}(\lambda, z) = \boldsymbol{Z}(\lambda, z)\boldsymbol{\Omega}(\lambda, z).$$

where

$$\boldsymbol{\Lambda}(\lambda, z) = \begin{pmatrix} 0 & \mathrm{i}z \\ 0 & 0 \end{pmatrix} + \frac{1}{4\lambda}\begin{pmatrix} V(z) & W(z) \\ 2 & -V(z) \end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix} X(z) & -2\mathrm{i}X(z)^2U(z) \\ -\mathrm{i}/(2U(z)) & -X(z) \end{pmatrix}$$

and

$$\boldsymbol{Z}(\lambda, z) = \lambda\begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix} + \frac{1}{4z}\begin{pmatrix} V(z) & W(z) \\ 2 & -V(z) \end{pmatrix} - \frac{1}{z\lambda}\begin{pmatrix} X(z) & -2\mathrm{i}X(z)^2U(z) \\ -\mathrm{i}/(2U(z)) & -X(z) \end{pmatrix}.$$

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$$\frac{\partial \boldsymbol{\Omega}}{\partial \lambda}(\lambda, z) = \boldsymbol{\Lambda}(\lambda, z)\boldsymbol{\Omega}(\lambda, z) \quad \text{and} \quad \frac{\partial \boldsymbol{\Omega}}{\partial z}(\lambda, z) = \boldsymbol{Z}(\lambda, z)\boldsymbol{\Omega}(\lambda, z).$$

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Tracking down our transformations gives

$$\lim_{\substack{n\to\infty \\ n \text{ even/odd}}} u_n(n^{-1}z) = U^{\mathrm{even/odd}}(z) \equiv U(z)$$

Compatibility Condition.

The system of equations

$$\frac{\partial \boldsymbol{\Omega}}{\partial \lambda}(\lambda, z) = \boldsymbol{\Lambda}(\lambda, z)\boldsymbol{\Omega}(\lambda, z) \quad \text{and} \quad \frac{\partial \boldsymbol{\Omega}}{\partial z}(\lambda, z) = \boldsymbol{Z}(\lambda, z)\boldsymbol{\Omega}(\lambda, z).$$

has compatibility condition

$$\frac{\partial \boldsymbol{\Lambda}}{\partial z}(\lambda, z) - \frac{\partial \boldsymbol{Z}}{\partial \lambda}(\lambda, z) + [\boldsymbol{\Lambda}(\lambda, z), \boldsymbol{Z}(\lambda, z)] = 0.$$

which implies

$$U''(z) - \frac{U'(z)^2}{U(z)} + \frac{U'(z)}{z} - \frac{4U(z)^2 + 4}{z} = 0.$$


We can breathe now.
Thank you!