Multivariate Bounded Rational Functions

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Midwestern Workshop on Asymptotic Analysis
October 11, 2024

One Variable: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

- Let p be a polynomial with no zeros on $\mathbb{D}$, so

$$p(z) = c(z - b_1)\cdots(z - b_n) \quad \text{for } b_1, \ldots, b_n \text{ with } |b_j| \geq 1$$

  and setting $\alpha_j = 1/\bar{b}_j$ gives

$$p(z) = d(1 - \bar{\alpha}_1 z)\cdots(1 - \bar{\alpha}_n z) \quad \text{for } \alpha_1, \ldots, \alpha_n \text{ with } |\alpha_j| \leq 1.$$

- $\tilde{p}(z) = z^n \overline{p(\frac{1}{\bar{z}})} = z^n \bar{d}(1 - \alpha_1 \frac{1}{z})\cdots(1 - \alpha_n \frac{1}{z}) = \bar{d}(z - \alpha_1)\cdots(z - \alpha_n)$.
- If $z \in \mathbb{T} = \partial\mathbb{D}$, each $|z - \alpha_j| = |1 - \bar{\alpha} z|$ so $|p(z)| = |\tilde{p}(z)|$.

Nice Rational Functions

Let ϕ be the following rational function with denominator p

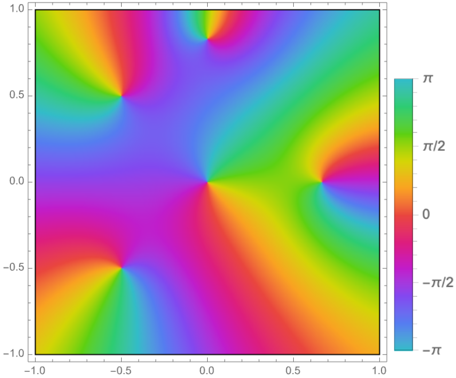
$$\phi(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{\bar{d}}{d}\left(\frac{z - \alpha_1}{1 - \bar{\alpha}_1 z}\right)\cdots\left(\frac{z - \alpha_n}{1 - \bar{\alpha}_n z}\right).$$

Then ϕ is holomorphic on $\mathbb{D}$ and $|\phi(z)| = 1$ a.e. on $\mathbb{T}$.

If $\alpha \in \mathbb{T}$, $(1 - \bar{\alpha}z) = (\bar{\alpha}\alpha - \bar{\alpha}z) = -\bar{\alpha}(z - \alpha)$.

A **finite Blaschke product** is a rational function of the form

$$\phi(z) = \eta z^m \left(\frac{z - \alpha_1}{1 - \bar{\alpha}_1 z} \right) \cdots \left(\frac{z - \alpha_n}{1 - \bar{\alpha}_n z} \right), \quad \text{for } \eta \in \mathbb{T}, \alpha_1, \ldots, \alpha_n \in \mathbb{D}.$$



Phase Portrait for ϕ with zeros: $0, -\frac{1}{2} \pm \frac{i}{2}, \frac{2}{3}, \frac{5i}{6}$.

Finite Blaschke Product Applications

- **L.U. Approximation**: Every holomorphic $f : \mathbb{D} \to \overline{\mathbb{D}}$ can be locally uniformly approximated by finite Blaschke products.

- **Nevanlinna-Pick Interpolation:** Given nodes $z^1, \ldots, z^n \in \mathbb{D}$ and $w^1, \ldots, w^n \in \mathbb{C}$, does there exist a holomorphic $f : \mathbb{D} \to \overline{\mathbb{D}}$ with $f(z^j) = w^j$?

  Every solvable NPI problem has finite Blaschke product solution.

- **Factoring functions** on $\mathbb{D}$: if $f \in H^p(\mathbb{D})$ then

  $$f = \text{Blaschke product} \cdot \text{Singular Inner Function} \cdot \text{Outer Function}$$

- **Shift-invariant subspaces** of the Hardy space $H^2(\mathbb{D})$.

  $$(Sf)(z) = zf(z), \text{ then } \phi H^2(\mathbb{D}) \text{ is closed and } S\text{-invariant.}$$

Reference: *Finite Blaschke Products and their Connections* by Stephan Garcia, Javad Mashreghi, William Ross.

Multivariable: $\mathbb{D}^d = \{z \in \mathbb{C}^d : \text{ each } |z_j| < 1\}$

- Let p be nonzero on $\mathbb{D}^d$ (stable) with multi-degree $m = (m_1, \dots, m_d)$.
- Define $\tilde{p}(z_1, \dots, z_d) := z_1^{m_1} \cdots z_d^{m_d} \overline{p(\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_d})}$.

Ex. $p(z) = 2 - z_1 - z_2 \quad \Rightarrow \quad \tilde{p}(z) = z_1 z_2 \left(2 - \frac{1}{z_1} - \frac{1}{z_2}\right) = 2z_1z_2 - z_2 - z_1.$

Then $\phi(z) = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$ satisfies $|\phi(z)| = 1$ a.e. on $\mathbb{T}^2$,

Rational Inner Functions (Rudin, Stout 1965)

Every rational ϕ holomorphic on $\mathbb{D}^d$ and with $|\phi(z)| = 1$ a.e. on $\mathbb{T}^d$ is of the form

$$\phi(z) = \eta z^n \frac{\tilde{p}(z)}{p(z)},$$

where $\eta \in \mathbb{T}$, $z^n = z_1^{n_1} \cdots z_d^{n_d}$, p is a polynomial with no zeros on $\mathbb{D}^d$, and p and $\tilde{p}$ have no common terms.

Rational Inner Functions on $\mathbb{D}^d$ and $\mathbb{D}^2$

- **L.U. Approximation**: Every holomorphic $f : \mathbb{D}^d \to \overline{\mathbb{D}}$ can be locally uniformly approximated by RIFs. (Rudin 1969)
- **Nevanlinna-Pick Interpolation**: Given nodes $z^1, \ldots, z^n \in \mathbb{D}^2$ and $w^1, \ldots, w^n \in \mathbb{C}$, does there exist a holomorphic $f : \mathbb{D}^2 \to \overline{\mathbb{D}}$ with $f(z^j) = w^j$?

  Every solvable Nevanlinna-Pick Interpolation problem has a RI solution ϕ. (Agler 1989)
- **RIFs & Stable Polynomials:** Polynomials p with no zeros on $\mathbb{D}^2$ have applications to systems and control engineering and other fields. (Kummert 1989, Ball-Sadosky-Vinnikov 2005)
  - Transfer function realizations
  - Sums of squares decompositions
- **Crucial Examples:** Matrix monotone functions, Clark measures, Schur-Agler class functions

Singularities of RIFs

Ex. (Cont.) Consider $\phi(z) = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$. Note that $\phi(1,1) = \frac{0}{0}$.

Note: Rational inner functions (RIFs) can have singularities on $\mathbb{T}^d$.

- Occur at common zeros of p and $\tilde{p}$
- When $d = 2$, there are finitely many of them. (depends on deg p)

Question: How do RIFs behave near their boundary singularities?

- On $\mathbb{T}^d$?
- On non-tangential approach regions inside $\mathbb{D}^d$?

We now restrict to $d = 2$.

"Factoring" stable polynomials when d=2

Let $\phi = \frac{\tilde{p}}{p}$ be a RIF with $p(1,1) = 0$ and let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Use the conformal map $\beta : \mathbb{H} \to \mathbb{D}$ with $\beta(z) = \frac{1+zi}{1-zi}$ to convert p stable on $\mathbb{D}^2$ with $p(1,1) = 0$ to P stable on $\mathbb{H}^2$ with $P(0,0) = 0$ via

$$P(z_1, z_2) = c(1 - z_1 i)^{m_1}(1 - z_2 i)^{m_2} p(\beta(z_1), \beta(z_2))$$
$$\Phi(z_1, z_2) = \phi(\beta(z_1), \beta(z_2)) = \frac{\overline{P}(z_1, z_2)}{P(z_1, z_2)}.$$

Ex. Let $p(z_1, z_2) = 2 - z_1 - z_2$. Then

$$\begin{aligned} P(z_1, z_2) &= c(1 - z_1 i)(1 - z_2 i)p(\beta(z_1), \beta(z_2)) = z_1 + z_2 - 2iz_1 z_2 \\ &= (1 - 2iz_1)\left(z_2 + \frac{z_1}{1 - 2iz_1}\right) \\ &= (1 - 2iz_1)\left(z_2 + z_1 + 2iz_1^2 - 4z_1^3 + O(z_1^4)\right). \end{aligned}$$

So near $(0,0)$, $\mathcal{Z}_P$ is parameterized by $z_2 = -\left(z_1 + 2iz_1^2 - 4z_1^3 + O(z_1^4)\right)$.

Puiseux Factorization

Recall $P(z_1, z_2) = (1 - 2iz_1)\left(z_2 + z_1 + 2iz_1^2 - 4z_1^3 + O(z_1^4)\right)$.

Puiseux Factorization (Knese 2015)

Let P be as above with $P(0,0) = 0$. Then

$$P(z) = u(z) \prod_{j=1}^{k} \prod_{m=1}^{M_j} \left(z_2 + q_j(z_1) + z_1^{2L_j} \psi_j^m \left(z_1^{1/M_j} \right) \right),$$

where each L_j is a positive integer, $q_j \in \mathbb{R}[z]$ with $q_j(0) = 0$, $\deg q_j < 2L_j$, and ψ_j^m is holomorphic near 0 with $\operatorname{Im}(\psi_j(0)) > 0$.

Ex (cont): $u(z) = 1 - 2iz_1$, $k = 1$, $M_1 = 1$, $q_1(z_1) = z_1$, $L_1 = 1$,
$\psi_1(z_1) = 2i - 4z_1 + O(z_1^2)$,

Another Example

Consider the polynomial p with no zeros on $\mathbb{D}^2$ and $p(1,1)=0$ given by

$$p(z_1,z_2) = 4 - 5z_1 - 2z_2 + 2z_1z_2 + 3z_1^2 - z_1^2z_2 - z_1^3z_2$$

Convert it to the polynomial with no zeros on $\mathbb{H}^2$ given by

$$P(z_1,z_2) = z_1 + z_2 - 2z_1^3 - 6z_1^2z_2 - i(z_1^2 + z_1z_2 - 4z_1^3z_2).$$

Then $P(0,0)=0$ and

$$\begin{aligned} P(z_1,z_2) &= \underbrace{(1 - iz_1 - 6z_1^2 + 4iz_1^3)}_{=:u(z)}\left(z_2 + z_1\frac{1 - iz_1 - 2z_1^2}{1 - iz_1 - 6z_1^2 + 4iz_1^3}\right) \\ &= u(z)(z_2 + z_1 + 4z_1^3 + 24z_1^5 + 8iz_1^6 + O(z_1^7)). \end{aligned}$$

For $x_1 \in \mathbb{R}$ and near 0, we have: $\{|\mathrm{Im}(z_2)| : P(x_1,z_2)=0\} \approx |x_1|^6$.

Key information from the Factorization

Contact Order

$K := \max\{2L_1, \cdots, 2L_k\}$ is the **contact order** of P at $(0,0)$, which measures how the zero set of P approaches $\mathbb{R}^2$ (along its quickest branch).

Universal Contact Order

$K_u := \min\{2L_1, \cdots, 2L_k\}$ is the **universal contact order** of P at $(0,0)$, which measures how the zero set of P approaches $\mathbb{R}^2$ (along its slowest branch).

Polynomial Truncation of P

$$[P](z) = \prod_{j=1}^{k} \left(z_2 + q_j(z_1) + i z_1^{2L_j} \right)^{M_j},$$

is the polynomial truncation of p, which extracts all "important" information about the zero set of P near $(0,0)$.

Question: How do RIFs ϕ or Φ behave near their boundary singularities?

- On $\mathbb{T}^2$ or $\mathbb{R}^2$?
- On non-tangential approach regions inside $\mathbb{D}^2$ or $\mathbb{H}^2$?

Contact Order

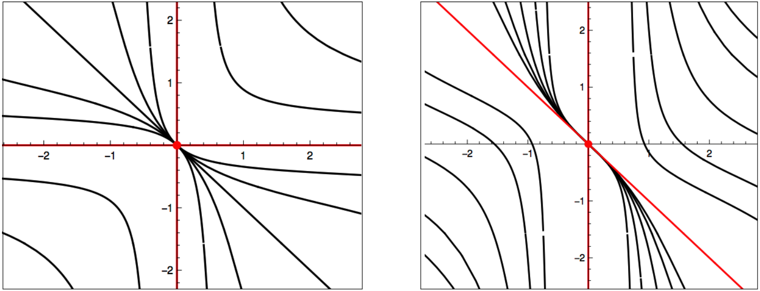


Figure 1. For two ϕs and some $\lambda \in \mathbb{T}$, the graphs of the level sets $\phi = \lambda$ on $\mathbb{T}^2$ represented as $[-\pi, \pi] \times [-\pi, \pi]$.

Theorem (B.-Pascoe-Sola 2020)

Then $\frac{\partial \phi}{\partial z_1}, \frac{\partial \phi}{\partial z_2} \in L^{\mathfrak{p}}_{\text{loc}}(\mathbb{T}^2)$ if and only if $\mathfrak{p} < \frac{1}{K} + 1$.

- If $\phi(z) = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}$, then K at $(1,1)$ is 2 and $\frac{\partial \phi}{\partial z_j} \in L^{\mathfrak{p}}(\mathbb{T}^2)$ iff $\mathfrak{p} < \frac{3}{2}$.

Universal Contact Order

Universal Contact Order

$K_u := \min\{2L_1, \cdots, 2L_k\}$ is the **universal contact order** of P at $(0,0)$.

For $z \in \mathbb{H}^2$, let $D_z = \{|z_1|, |z_2|, \mathrm{Im}(z_1), \mathrm{Im}(z_2)\}$
For $c \geq 1$, the associated non-tangential approach region to 0 in $\mathbb{H}^2$ is:

$$AR_c = \left\{z \in \mathbb{H}^2 : c \geq x/y \geq 1/c \text{ for any } x, y \in D_z\right\}$$

Theorem (B.-Knese-Pascoe-Sola, 2021)

$\Phi(z) = \frac{\bar{P}(z)}{P(z)}$ has a non-tangential polynomial approximation to order $K_u - 2$.
Specifically, there is a polynomial Q of degree at most $K_u - 2$ so that

$$\Phi(z) = Q(z) + O\left(|z_1|^{K_u - 1}\right)$$

on any non-tangential region AR_c as $|z_1| \to 0$.

Example (Cont.)

Consider $P(z_1, z_2) = z_1 + z_2 - 2z_1^3 - 6z_1^2 z_2 - i(z_1^2 + z_1 z_2 - 4z_1^3 z_2)$.

Recall that: $P(z_1, z_2) = u(z)(z_2 + z_1 + 4z_1^3 + 24z_1^5 + 8iz_1^6 + O(z_1^7))$.

Then $K_u = 6$ and $K_u - 2 = 4$. so Φ has a non-tangential polynomial approximation to order 4 near 0.

One can show that on non-tangential approach regions to 0,

$$\Phi(z_1, z_2) = 1 + 2iz_1 - 2z_1^2 + 2iz_1^3 - 6z_1^4 - \frac{2iz_1^5(z_1 - 7z_2)}{z_1 + z_2} + O(|z_1|^6)$$

Question: What are the bounded rational functions on $\mathbb{D}^d$ or $\mathbb{H}^d$?

Admissible Numerator Question

Note: If p is the denominator of a finite Blaschke product, then p has no zeros on $\overline{\mathbb{D}}$.

Thus, q/p is bounded on $\overline{\mathbb{D}}$ for all polynomials q.

Let p be the denominator of a rational inner function on $\mathbb{H}^2$.

Question: If $p(0,0) = 0$, what are the functions q so that $\frac{q}{p}$ is bounded and holomorphic in $\mathbb{H}^2$ intersected with a neighborhood of 0?

Def. We call these $\frac{q}{p}$ locally H^∞.

Note: Let $R_0 = \mathbb{C}\{x, y\}$, the set of convergent power series centered at 0.

Any polynomial $q \in (p, \bar{p})R_0$ gives $\frac{q}{p}$ bounded in $\mathbb{H}^2$.

Admissible Numerators: Vanishing Order 1

Theorem (B.-Knese-Pascoe-Sola 2021

Let p be a stable polynomial on $\mathbb{H}^2$ such that p vanishes to order 1 at $(0,0)$. Then if $q \in \mathbb{C}[z_1, z_2]$, then q/p is locally H^∞ if and only if $q \in (p, \bar{p})R_0$.

Ex. Consider the polynomial with no zeros on $\mathbb{H}^2$ given by

$$\begin{aligned} p(z_1, z_2) &= z_1 + z_2 - 2z_1^3 - 6z_1^2 z_2 - i(z_1^2 + z_1 z_2 - 4z_1^3 z_2) \\ &= u(z)(z_2 + z_1 + 4z_1^3 + 24z_1^5 + 8iz_1^6 + O(z_1^7)). \end{aligned}$$

Then p clearly vanishes to order 1 at 0 and so for a polynomial q, $\frac{q}{p}$ is locally H^∞ if and only if

$$q \in (p, \bar{p})R_0 = (z_2 + z_1 + 4z_1^3 + 24z_1^5, z_1^6)R_0$$

Admissible Numerators: Additional cases

Recall that $[p](z) = \prod_{j=1}^{k} (z_2 + q_j(z_1) + i z_1^{2L_j})^{M_j}$.

Define product ideal $\mathcal{I} = \prod_{j=1}^{k} (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0$.

Theorem (B.-Knese-Pascoe-Sola 2021)

Let p be a stable polynomial on $\mathbb{H}^2$ with $p(0,0) = 0$ and $q \in \mathbb{C}[z_1, z_2]$.

- If $f \in \mathcal{I}$, then q/p is locally H^∞.
- Suppose p either has a double point, an ordinary multiple point, or repeated segments (all q_j are the same). If q/p is locally H^∞ then $q \in \mathcal{I}$.

Full Numerator Criterion (Conjecture)

For $q \in \mathbb{C}[z_1, z_2]$, q/p is locally H^∞ if and only if $q \in \mathcal{I}$.

Ex. Consider the polynomial

$$p(z_1, z_2) = -4(z_2^2 + 4z_1 z_2 + z_1^2 - 2z_1^2 z_2^2 - 4iz_1 z_2(z_1 + z_2))$$
$$= \underbrace{-4(1 - 4iz_1 - 2z_1^2)}_{=:u} \left(z_2^2 + \frac{4z_1(1 - iz_1)}{1 - 4iz_1 - 2z_1^2} z_2 + \frac{z_1^2}{1 - 4iz_1 - 2z_1^2} \right).$$

Near $(0, 0)$ we can factor p/u as:

$$\left(z_2 + z_1 \frac{(2(1-iz_1)+\sqrt{3-4iz_1-2z_1^2})}{1-4iz_1-2z_1^2} \right) \left(z_2 + z_1 \frac{(2(1-iz_1)-\sqrt{3-4iz_1-2z_1^2})}{1-4iz_1-2z_1^2} \right)$$
$$= (z_2 + (2+\sqrt{3})z_1 + i\left(6 + \frac{10}{\sqrt{3}}\right) z_1^2 + \cdots)(z_2 + (2-\sqrt{3})z_1 + i\left(6 - \frac{10}{\sqrt{3}}\right) z_1^2 + \cdots)$$
$$= (z_2 + q_1(z_1) + icz_1^2 + \cdots)(z_2 + q_2(z_1) + idz_1^2 + \cdots)$$

Then $\mathcal{I} = (z_2 + q_1(z_1), z_1^2)(z_2 + q_2(z_1), z_1^2)R_0$, with generators:

$$(z_2 + (2+\sqrt{3})z_1)(z_2 + (2-\sqrt{3})z_1), z_1^2(z_2 + (2-\sqrt{3})z_1), z_1^2(z_2 + (2+\sqrt{3})z_1), z_1^4$$

Full Numerator Criterion (Kollár 2024, Knese 2024)

For $q \in \mathbb{C}[z_1, z_2]$, q/p is locally H^∞ if and only if $q \in \mathcal{I}$.

Question: What about polynomials p stable in $\mathbb{D}^d$ or $\mathbb{H}^d$?

Admissible Numerators in Product Domains

First Case: Assume p is stable on $\mathbb{H}^{d+1}$ with variables $(x_1, \dots, x_d, z)$, $p(0) = 0$, the zero 0 of p is isolated with respect to $\mathbb{R}^{d+1}$, and $\frac{\partial p}{\partial z}(0) \neq 0$, so p factors as

$$p(x, z) = u(x, z)\,(z + \phi(x))$$

for u, ϕ holomorphic near 0 with $u(0, 0) \neq 0$ and $\phi(0) = 0$.

Lemma (B.-Knese-Pascoe-Sola 2024)

The function ϕ above must have the form

$$\phi(x) = \phi_1(x) + \cdots + \phi_{2L-1}(x) + \phi_{2L}(x) + \cdots$$

where each ϕ_j is a homogeneous polynomial of degree j, $\phi_1, \dots, \phi_{2L-1} \in \mathbb{R}[x_1, \dots, x_d]$ have real coefficients, and $\operatorname{Im}(\phi_{2L}) \not\equiv 0$ and is non-negative on $\mathbb{R}^d$.

Admissible Numerators

Ex. Let p be stable on $\mathbb{H}^3$ given by

$$\begin{aligned} p(x_1, x_2, z) &= x_1 + x_2 + z - 2i(x_1x_2 + x_1z + x_2z) - 3x_1x_2z \\ &= (1 - 2i(x_1 + x_2) - 3x_1x_2)\left(z + \frac{x_1 + x2 - 2ix_1x_2}{1 - 2i(x_1 + x_2) - 3x_1x_2}\right) \end{aligned}$$

where $\phi(x_1, x_2) = x_1 + x_2 + 2i(x_1^2 + x_1x_2 + x_2^2) +$ higher order terms.

Theorem (B.-Knese-Pascoe-Sola 2024)

Let $\operatorname{Im}(\phi_{2L})$ be positive definite. For $q \in \mathbb{C}[x, z]$, q/p is locally H^∞ near 0 if and only if

$$q \in \mathcal{I} = (z + r(x), (x)^{2L})R_0$$

where $r(x) = \sum_{j<2L} \phi_j(x)$ and $(x)^{2L}$ is the set of powers x^α for $|\alpha| = 2L$.

Ex. (Cont.) $\mathcal{I} = (x_1 + x_2 + z, x_1^2, x_1x_2, x_2^2)R_0$

Question Are there p with $\mathrm{Im}(\phi_{2L})$ positive semi-definite but not positive-definite on $\mathbb{R}^d$?

Ex. Start with p_0 with no zeros on $\mathbb{D}^2$ with $p(1,1) = 0$ given by

$$p_0(x, y) = x^2 - xy - 3x - y + 4.$$

Then, consider $p_1(x, y, z) = p_0((x + y)/2, z)$ Converting this to a stable polynomial on $\mathbb{H}^3$ gives

$$\begin{aligned} p(x, y, z) = {} & 2x^2y^2z + 2i\,x^2y^2 + 3i\,x^2yz + 3i\,xy^2z - \tfrac{5}{2}\,x^2y \\ & - \tfrac{5}{2}\,xy^2 - \tfrac{5}{4}\,x^2z - \tfrac{9}{2}\,xyz - \tfrac{5}{4}\,y^2z - \tfrac{3}{4}i\,x^2 \\ & - \tfrac{5}{2}i\,xy - \tfrac{3}{4}i\,y^2 - 2i\,xz - 2i\,yz + \tfrac{1}{2}\,x + \tfrac{1}{2}\,y + z. \end{aligned}$$

And, $\phi(x, y) = \frac{1}{2}(x + y) + \frac{i}{4}(x - y)^2 +$ higher order terms.

Letting $H(x, y) = \frac{1}{2}(x + y) + \frac{1}{8}(x^3 + 7x^2y + 7xy^2 + y^3)$, we can find that for $q \in \mathbb{C}[x, z]$, q/p is locally H^∞ if and only if

$$q \in \mathcal{I} = (z + H(x, y), (x - y)^2, (x - y)(x + y)^2, (x, y)^4).$$

The Last Slide

Takeaways

- RIFs and bounded rational functions are straightforward on $\mathbb{D}$.
- RIFs on $\mathbb{D}^2$ and $\mathbb{H}^2$ have well-understood boundary regularity properties.
- (Locally) Bounded rational functions on $\mathbb{D}^2$ and $\mathbb{H}^2$ have characterizations.
- In higher dimensions, there are many open questions about the structure and properties of rational inner functions and more generally, bounded rational functions.

Many thanks to the organizers of MWAA!!