Large gap asymptotics for the Bessel kernel determinant

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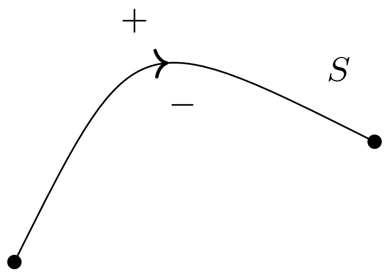
Collaborators

This is joint work with Christophe Charlier (Lund University) and Jonatan Lenells (KTH Royal Institute of Technology).



Given a smooth, oriented curve S in the complex plane and a Hölder continuous function $\phi(z)$ on S, find a function $\Psi(z)$, analytic on $\mathbb{C} \setminus S$, which satisfies

$$\Psi_+(z) - \Psi_-(z) = \phi(z), \quad z \in S.$$

A solution is given by the **Sokhotski-Plemelj formula**:

$$\Psi(z) = \frac{1}{2\pi i} \int_S \frac{\phi(w)}{w - z} dw.$$

What is the Bessel kernel determinant?

Let $\mathcal{K}|_{\mathcal{I}}$ be the trace class operator with the kernel

$$K(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}$$

acting on $L^2(\mathcal{I})$, where J_α is the Bessel function of the first kind with order $\alpha > -1$, and $\mathcal{I} \subseteq \mathbb{R}$. The object of study is the Fredholm determinant

$$F(\mathcal{I}) = \det(I - \mathcal{K}|_{\mathcal{I}}),$$

which represents a *gap probability* for the Bessel point process. Let

$$\mathcal{I}_g := (0, x_1) \cup (x_2, x_3) \cup \cdots \cup (x_{2g}, x_{2g+1}).$$

We study the gap probability $F(r\mathcal{I}_g)$ in the limit $r \to +\infty$, i.e. we wish to obtain the large gap asymptotics.

Results for $g = 0$

Theorem (Tracy, Widom '94)

Let $\mathcal{I}_0 = (0, x_1)$ *and* $\alpha > -1$. *As* $r \to +\infty$,

$$F(r\mathcal{I}_0) = \exp\left(-\frac{rx_1}{4} + \alpha\sqrt{rx_1} - \frac{\alpha^2}{4}\log r + C_0 + \mathcal{O}(r^{-\frac{1}{2}})\right),$$

where α *and* C_0 *are independent of* r.

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where α *and* C_0 *are independent of* r.

Theorem (Ehrhardt $\alpha \in (-1, 1)$, '10, Deift, Krasovsky, Vasilevska $\alpha > -1$, '11)

The constant C_0 *is given by*

$$C_0 = G(1+\alpha)(2\pi)^{-\frac{\alpha}{2}} - \frac{\alpha^2}{4}\log x_1,$$

where G *denote the Barnes' G-function.*

Differential Identity

$\mathcal{K}|_{\mathcal{I}_1}$ is an operator with an integrable kernel (in the sense of Its et al.)! This means its resolvent kernel can be expressed in terms of the solution of a RHP.

Theorem (B., Charlier, Lenells '23)

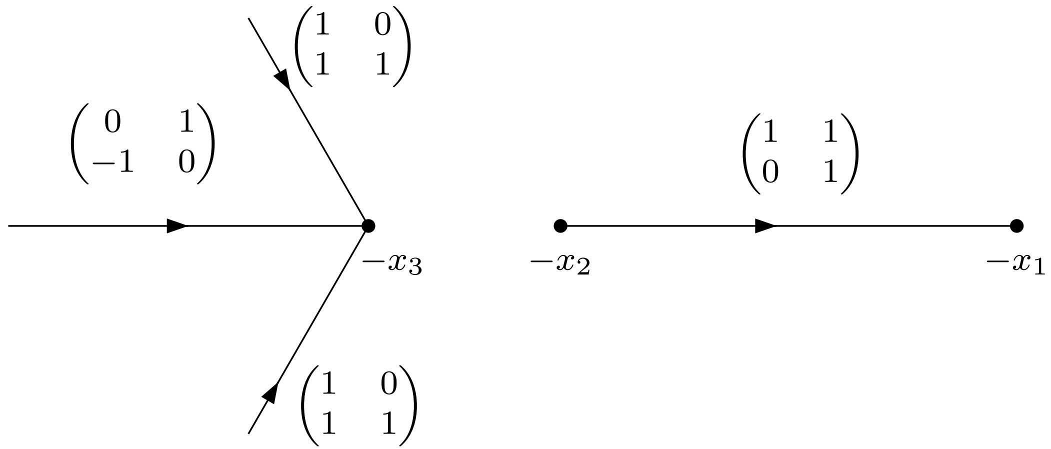
Let $0 < x_1 < x_2 < x_3 < +\infty$ be fixed. We have the identity

$$\partial_r \log F(r\mathcal{I}_1) = \tfrac{1}{2ir}\Phi_{1,12}(r) + \tfrac{1}{16r},$$

where $\Phi_1(r) = \Phi_1(r;\vec{x})$ is defined by

$$\Phi_1(r) = \lim_{z\to\infty} rz\left(\Phi(z)e^{-\sqrt{rz}\sigma_3}M^{-1}(rz)^{\frac{\sigma_3}{4}} - I\right), \quad M = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and $\Phi(\cdot) = \Phi(\cdot; r, \vec{x})$ is the unique solution of the following RH problem.



RH problem for $\Phi(\cdot) = \Phi(\cdot; r, \vec{x})$

(a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \to \mathbb{C}^{2\times 2}$ is analytic, where the contour Σ_Φ is shown in the next slide.

(b) We have the jump conditions

$$\Phi_+(z) = \Phi_-(z) J_\Phi(z), \qquad\qquad z \in \Sigma_\Phi.$$

(c) As $z \to \infty$, we have

$$\Phi(z) = \Big(I + \mathcal{O}(z^{-1})\Big)(rz)^{-\frac{\sigma_3}{4}} M e^{\sqrt{rz}\sigma_3},$$

where the principal branch is chosen for each fractional power.

(d) As $z \to -x_j$, $j = 1, 2, 3$, we have $\Phi(z) = \mathcal{O}(\log(z + x_j))$.

Normalize Φ at $z = \infty$

To normalize $\Phi(z)$ at $z = \infty$, we introduce a $\mathfrak{g}$-function, so it should have the behavior

$$\mathfrak{g}(z) = \sqrt{z}(1 + \mathcal{O}(z^{-1})), \quad z \to \infty.$$

The idea is to define a new function

$$\tilde{T}(z) = \Phi(z) e^{-\sqrt{r}\mathfrak{g}(z)\sigma_3}.$$

We see that $\mathfrak{g}(z)$ has a branch cut at $z = \infty$. Let's try to choose the jumps of $\mathfrak{g}(z)$ to our advantage. Now we can compute the jumps of $\tilde{T}(z)$; for example, when $z \in (-\infty, -x_3)$,

$$\begin{aligned}\tilde{T}_+(z) &= \Phi_+(z) e^{-\sqrt{r}\mathfrak{g}_+(z)\sigma_3} \\ &= \Phi_-(z) e^{-\sqrt{r}\mathfrak{g}_-(z)\sigma_3} e^{\sqrt{r}\mathfrak{g}_-(z)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\sqrt{r}\mathfrak{g}_+(z)\sigma_3} \\ &= \tilde{T}_-(z) \begin{pmatrix} 0 & e^{\sqrt{r}(\mathfrak{g}_+(z) - \mathfrak{g}_-(z))\sigma_3} \\ e^{-\sqrt{r}(\mathfrak{g}_+(z) - \mathfrak{g}_-(z))\sigma_3} & 0 \end{pmatrix}.\end{aligned}$$

Choosing the jumps of $\mathfrak{g}(z)$

Assuming $\mathfrak{g}(z)$ has jumps only on $(-\infty,-x_1)$, we find that $\tilde{T}(z)$ has the jumps

$$\tilde{T}_+(z)=\tilde{T}_-(z)\begin{cases}\begin{pmatrix}0 & e^{\sqrt{r}(\mathfrak{g}_+(z)+\mathfrak{g}_-(z))}\\ e^{-\sqrt{r}(\mathfrak{g}_+(z)+\mathfrak{g}_-(z))} & 0\end{pmatrix}, & z\in(-\infty,-x_3),\\ e^{-\sqrt{r}(\mathfrak{g}_+(z)-\mathfrak{g}_-(z))\sigma_3}, & z\in(-x_3,-x_2),\\ \begin{pmatrix}e^{-\sqrt{r}(\mathfrak{g}_+(z)-\mathfrak{g}_-(z))} & e^{\sqrt{r}(\mathfrak{g}_+(z)+\mathfrak{g}_-(z))}\\ 0 & e^{\sqrt{r}(\mathfrak{g}_+(z)-\mathfrak{g}_-(z))}\end{pmatrix}, & z\in(-x_2,-x_1).\end{cases}$$

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Thus, let's determine $\mathfrak{g}(z)$ by the conditions

1. $\mathfrak{g}(z)$ is analytic for $z\in\mathbb{C}\setminus(-\infty,-x_1)$,
2. $\mathfrak{g}(z)=\sqrt{z}(1+\mathcal{O}(z^{-1}))$ as $z\to\infty$,
3. $\mathfrak{g}(z)$ has the jump conditions

$$\begin{aligned}\mathfrak{g}_+(z)+\mathfrak{g}_-(z)&=0, && z\in(-\infty,-x_3)\cup(-x_2,-x_1),\\ \mathfrak{g}_+(z)-\mathfrak{g}_-(z)&=i\Omega, && z\in(-x_3,-x_2),\end{aligned}$$

   where Ω is a constant.

Let's use the Sokhotski–Plemelj formula to determine $\mathfrak{g}(z)$! Differentiating the jump conditions for $\mathfrak{g}(z)$, we have

$$\begin{aligned} \mathfrak{g}'_+(z) + \mathfrak{g}'_-(z) &= 0, && z \in (-\infty, -x_3) \cup (-x_2, -x_1), \\ \mathfrak{g}'_+(z) - \mathfrak{g}'_-(z) &= 0, && z \in (-x_3, -x_2). \end{aligned}$$

Define $\sqrt{\mathcal{R}(z)} := \sqrt{(z+x_1)(z+x_2)(z+x_3)}$ with $\mathcal{R}(z) > 0$ for $z > -x_1$ and jumps

$$\sqrt{\mathcal{R}(z)}_+ + \sqrt{\mathcal{R}(z)}_- = 0, \quad z \in (-\infty, -x_3) \cup (-x_2, -x_1).$$

Now notice that

$$\left(\mathfrak{g}'(z)\sqrt{\mathcal{R}(z)}\right)_+ - \left(\mathfrak{g}'(z)\sqrt{\mathcal{R}(z)}\right)_- = 0, \quad z \in (-\infty, -x_1).$$

For $\mathfrak{g}(z)$ to have the correct behavior at $z = \infty$, we must have

$$\mathfrak{g}'(z) = \frac{q_1 z + q_0}{\sqrt{\mathcal{R}(z)}}.$$

$\mathfrak{g}(z)$ function

$$\mathfrak{g}(z) = \int_{-x_1}^{z} \frac{\frac{s}{2} + q_0}{\sqrt{\mathcal{R}(s)}} ds, \quad \text{where } q_0 \text{ is defined by } \int_{-x_3}^{-x_2} \frac{\frac{s}{2} + q_0}{\sqrt{\mathcal{R}(s)}} ds = 0.$$

The $\mathfrak{g}$-function has the following properties:

1. The $\mathfrak{g}$-function is analytic in $\mathbb{C} \setminus (-\infty, -x_1]$ and satisfies $\mathfrak{g}(z) = \overline{\mathfrak{g}(\overline{z})}$.
2. The $\mathfrak{g}$-function satisfies the jump conditions

$$\begin{aligned} \mathfrak{g}_+(z) + \mathfrak{g}_-(z) &= 0, && z \in (-\infty, -x_3) \cup (-x_2, -x_1), \\ \mathfrak{g}_+(z) - \mathfrak{g}_-(z) &= i\Omega, && z \in (-x_3, -x_2), \end{aligned}$$

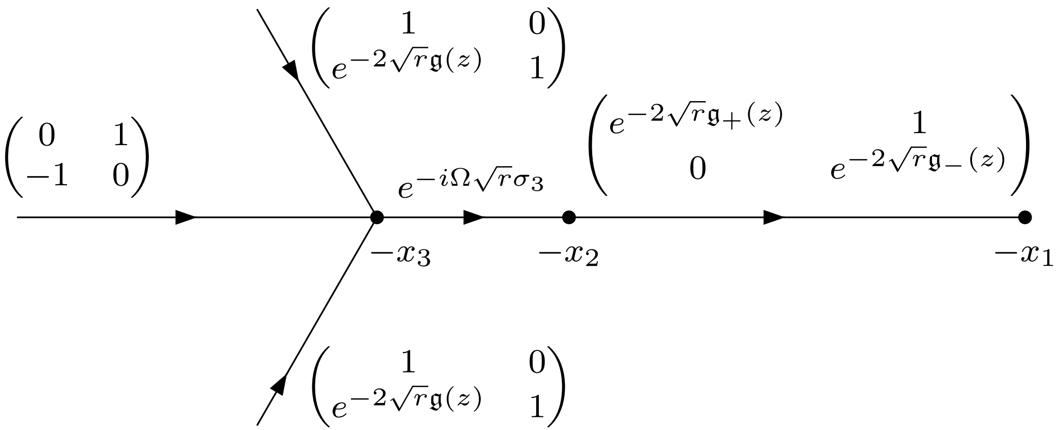
   where $\Omega = 2\int_{-x_2}^{-x_1} \frac{\frac{s}{2}+q_0}{|\mathcal{R}(s)|^{\frac{1}{2}}} ds > 0$.
3. As $z \to \infty$, $z \notin (-\infty, -x_3)$, we have

$$\mathfrak{g}(z) = \sqrt{z} - \frac{2c}{\sqrt{z}} + \mathcal{O}(z^{-3/2}), \quad c := q_0 - \frac{x_1 + x_2 + x_3}{4}.$$

4. $\operatorname{Re} \mathfrak{g}(z) \geq 0$ for $z \in \mathbb{C}$ with equality only when $z \in (-\infty, -x_3] \cup [-x_2, -x_1]$.

Normalize $\Phi(z)$ at $z = \infty$ by defining $T(z) := \begin{pmatrix} 1 & 0 \\ -2ic\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} \Phi(z) e^{-\sqrt{r}\mathfrak{g}(z)\sigma_3}$.

RHP for T



RH problem for $T(\cdot) = T(\cdot; r, \vec{x})$

(a) $T : \mathbb{C} \backslash \Sigma_T \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for T are given by

$$T_+(z) = T_-(z) J_T(z), \qquad z \in \Sigma_T.$$

(c) As $z \to \infty$, we have

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}\left(z^{-2}\right) \right) z^{-\frac{\sigma_3}{4}} M.$$

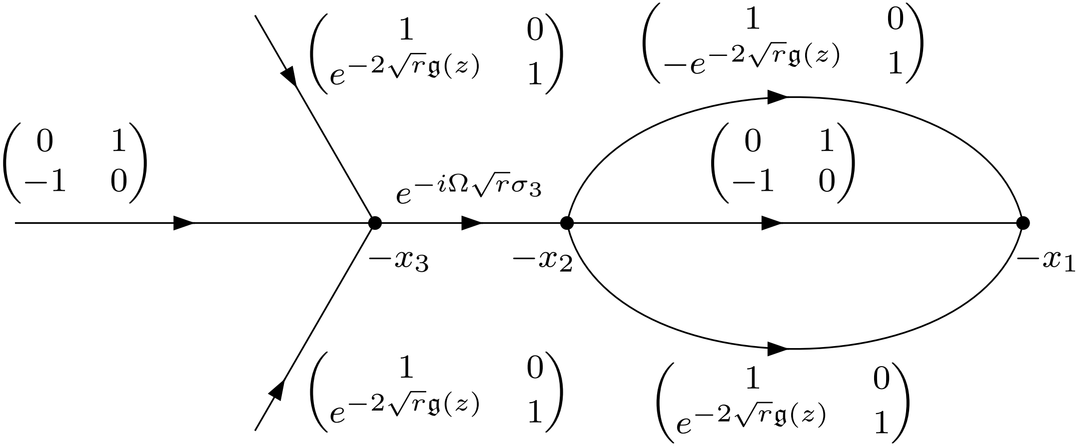
(d) $T(z) = \mathcal{O}(\log(z + x_j))$ as $z \to -x_j$, $j = 1, 2, 3$.

Notice that

$$\begin{pmatrix} e^{-2\sqrt{r}\mathfrak{g}_+(z)} & 1 \\ 0 & e^{-2\sqrt{r}\mathfrak{g}_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-2\sqrt{r}\mathfrak{g}_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2\sqrt{r}\mathfrak{g}_+(z)} & 1 \end{pmatrix}.$$

We now open the ‘lenses’ by defining the new matrix

$$S(z) := T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-2\sqrt{r}\mathfrak{g}(z)} & 1 \end{pmatrix}, & z \in \mathcal{L} \text{ and } \operatorname{Im} z > 0, \\ \begin{pmatrix} 1 & 0 \\ e^{-2\sqrt{r}\mathfrak{g}(z)} & 1 \end{pmatrix}, & z \in \mathcal{L} \text{ and } \operatorname{Im} z < 0, \\ I, & \text{otherwise.} \end{cases}$$



RH problem for $S(\cdot) = S(\cdot; r, \vec{x})$

(a) $S : \mathbb{C}\backslash\Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for S are given by

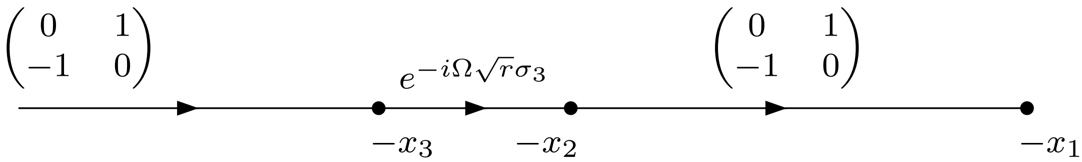
$$S_+(z) = S_-(z)J_S(z), \qquad z \in \Sigma_S.$$

(c) As $z \to \infty$, we have

$$S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}\left(z^{-2}\right)\right) z^{-\frac{\sigma_3}{4}} M.$$

(d) $S(z) = \mathcal{O}(\log(z + x_j))$ as $z \to -x_j$, $j = 1, 2, 3$.

Global parametrix $P^{(\infty)}$



RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \backslash (-\infty, -x_1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -x_3) \cup (-x_2, -x_1),$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{-i\Omega\sqrt{r}\sigma_3}, \qquad z \in (-x_3, -x_2),$$

(c) As $z \to \infty$, we have

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}\left(z^{-2}\right) \right) z^{-\frac{\sigma_3}{4}} M. \tag{1}$$

(d) As $z \to -x_j$, $j = 1, 2, 3$, we have $P^{(\infty)}(z) = \mathcal{O}((z + x_j)^{-\frac{1}{4}})$.

This RHP is explicitly solvable in terms of Jacobi θ-functions!

Local Parametrices $P^{(-x_j)}$

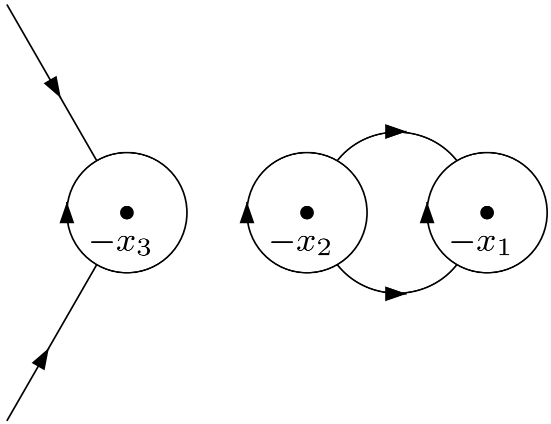
Let $\mathcal{D}_j$ be a small neighborhood of $-x_j$, $j = 1, 2, 3$. We approximate $S(z)$ with a local parametrix $P^{(-x_j)}(z)$ so that:

1. $P^{(-x_j)}(z)$ has the (exact) same jumps as $S(z)$ for $z \in \mathcal{D}_j$.
2. $S(z)P^{(-x_j)}(z)^{-1} = \mathcal{O}(1)$ as $z \to -x_j$.
3. $P^{(-x_j)}$ "matches" with $P^{(\infty)}$, in the sense that

$$P^{(-x_j)}(z) = (I + o(1))P^{(\infty)}(z) \qquad \text{as } r \to +\infty,$$

uniformly for $z \in \partial\mathcal{D}_j$.

The local parametrices $P^{(-x_j)}(z)$ can be explicitly constructed in terms of Bessel functions!



Define the error matrix

$$R(z) := \begin{cases} S(z)P^{(-x_j)}(z)^{-1}, & z \in \mathcal{D}_j,\ j = 1,2,3, \\ S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \mathcal{D}_j. \end{cases}$$

The jumps of $R(z)$ on Σ_R is denoted $J_R(z)$ and has the properties

$$J_R(z) = \begin{cases} I + \mathcal{O}(e^{-\tilde{c}|rz|^{\frac{1}{2}}}) & \text{as } r \to +\infty \text{ uniformly for } z \in \Sigma_R \setminus \left(\cup_{j=1}^{3} \partial\mathcal{D}_j \right), \\ I + \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}) & \text{as } r \to +\infty \text{ uniformly for } z \in \cup_{j=1}^{3} \partial\mathcal{D}_j. \end{cases}$$

It follows from the small-norm theory of RHPs that

$$R(z) = I + \frac{R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}) \qquad \text{as } r \to +\infty \text{ uniformly for } z \in \mathbb{C} \setminus \Sigma_R.$$

Putting the pieces together

We can now unravel our transformations to get the expression

$$\Phi(z) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ 2ic\sqrt{r} & 1 \end{pmatrix} R(z) P^{(\infty)}(z) e^{\sqrt{r}\mathfrak{g}(z)\sigma_3}.$$

Now we can compute the residue and obtain

$$\frac{\Phi_{1,12}(r)}{2ir} = c + \frac{P^{(\infty)}_{1,12}}{2i\sqrt{r}} + \frac{R^{(\frac{1}{2})}_{1,12}}{2ir} + \mathcal{O}(r^{-\frac{3}{2}}) \qquad \text{as } r \to +\infty,$$

where $R^{(\frac{1}{2})}_{1,12}$ is obtained from $R^{(1)}(z)$.

Lemma

As $r \to +\infty$,

$$\frac{\Phi_{1,12}(r)}{2ir} = \frac{d}{dr}\left[cr + \log\theta(-\tfrac{\Omega\sqrt{t}}{2\pi}) - \frac{1}{32}\sum_{j=1}^{3}\int_M^r \mathcal{B}(-x_j, -\tfrac{\Omega\sqrt{t}}{2\pi})\frac{dt}{t}\right] + \mathcal{O}(r^{-\frac{3}{2}}),$$

where $M > 0$ is independent of r and $\mathcal{B}$ is a ratio of θ-functions.

Result for $g = 1$, $\alpha = 0$

The integrals of $\mathcal{B}$ can be computed using properties of θ-functions. We find that

$$\int_M^r \mathcal{B}(-x_j, -\tfrac{\Omega\sqrt{t}}{2\pi})\frac{dt}{t} = 2\log(r) + \tilde{C}_j + \mathcal{O}(r^{-1}).$$

Thus, we reach our result:

Theorem

Let $g = 1$, $\alpha = 0$ and fix $0 < x_1 < x_2 < x_3 < +\infty$. As $r \to +\infty$,

$$F(r\mathcal{I}_1) = \exp\left(c\, r - \frac{1}{8}\log r + \log\theta\left(-\tfrac{\Omega\sqrt{r}}{2\pi}\right) + C + \mathcal{O}(r^{-\frac{1}{2}})\right),$$

where C is independent of r.

Result for $g \geq 0$, $\alpha > -1$

Theorem (B., Charlier, Lenells '23)

Let $0 < x_1 < x_2 < \cdots < x_{2g+1} < \infty$ and $\alpha > -1$ be fixed. Then, for almost all choices of $x_1, x_2, \ldots, x_{2g+1}$, as $r \to +\infty$,

$$F(r\mathcal{I}_g) = \exp\left(c\, r - d_1(\alpha)\sqrt{r} - \frac{g + 2\alpha^2}{8} \log r + \log \Theta(\vec{\nu}(r)) + C + \mathcal{O}(r^{-\frac{1}{2}}) \right),$$

where C is independent of r and $\Theta(\cdot)$ is the Riemann Θ-function. For $g = 0$, we understand that $\Theta(\cdot) \equiv 1$.

The integral term in the $g > 1$ case is significantly more challenging. One must understand the winding of a g-dimensional torus. A novelty was the use of Birkoff's ergodic theorem.

References

E. Blackstone, C. Charlier, and J. Lenells, The Bessel kernel determinant on large intervals and Birkhoff's ergodic theorem. *Comm. Pure Appl. Math.* **76** (2023), 3300–3345.

P. Deift, I. Krasovsky, and J. Vasilevska, Asymptotics for a determinant with a confluent hypergeometric kernel, *Int. Math. Res. Not.* **9** (2011), 2117–2160.

P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* **137** (1993), 295–368.

A. Its, A. Izergin, V. Korepin, and N. Slavnov, Differential equations for quantum correlation functions. Proceedings of the Conference on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory, 1990.

C. Tracy and H. Widom, Level-spacing distributions and the Bessel kernel. *Comm. Math. Phys.* **161** (1994), 289–309.