Weighted Bergman kernels, weak-type estimates, and Schur's test on non-smooth domains in $\mathbb{C}^n$

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The Bergman space

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$$L^2(\Omega) := \left\{ f : \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\}.$$

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By the mean value property for holomorphic functions, for $f \in A^2(\Omega)$ and $K \subset \Omega$ compact,

$$\sup_K |f(z)| \leq c \|f\|_{L^2(\Omega)}.$$

Hence $A^2(\Omega)$ is a closed Hilbert subspace of $L^2(\Omega)$.

The Bergman kernel

Since the evaluation functional is a bounded linear functional on $A^2(\Omega)$, the Riesz representation theorem ensures that for each $z \in \Omega$ and each $f \in A^2(\Omega)$ there is a function $\mathbf{B}_\Omega(z, w)$ that is holomorphic in z and antiholomorphic in w such that

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Studying the mapping properties of this integral operator and its kernel is a well-known topic in several complex variables!

Mapping properties of the Bergman projection

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The Bergman projection is bounded on $L^p(\Omega)$ for all $p \in (1, \infty)$ on

- strongly pseudoconvex domains (Fefferman, Phong-Stein 70's)
- convex domains of finite-type in $\mathbb{C}^n$ (McNeal-Stein '86

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On $\mathbb{D} = \{|z| < 1\}$,

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Theorem

The Bergman projection on $\mathbb{D}$ is bounded on $L^p(\mathbb{D})$ for all $p \in (1, \infty)$. In fact, the same range of boundedness holds for the operator

$$B_\alpha f(z) = \int_{\mathbb{D}} \frac{|w|^\alpha}{|1 - z\bar{w}|^2} |f(w)| \, dV(w),$$

provided that $\alpha > -2$.

Method of proof: Schur's lemma

Lemma

Let $\Omega \subset \mathbb{C}^n$ be a domain, K an a.e. positive measurable function on $\Omega \times \Omega$ and $\mathbf{K}$ the associated integral operator to K. Suppose there exists a positive auxiliary function h on Ω and $0 < a < b$ such that for all $\varepsilon > 0$ the following hold

$$\mathbf{K}(h^{-\varepsilon})(z) = \int_\Omega K(z,w)h(w)^{-\varepsilon}\, dV(w) \lesssim h(z)^{-\varepsilon};$$

$$\mathbf{K}(h^{-\varepsilon})(w) = \int_\Omega K(z,w)h(z)^{-\varepsilon}\, dV(z) \lesssim h(w)^{-\varepsilon}.$$

Then $\mathbf{K}$ is bounded on $L^p(\Omega)$ for all $p \in \left(\frac{a+b}{b}, \frac{a+b}{a}\right)$.

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Remark

In practice, the function h vanishes on the set where $K(z,w)$ is singular; hence $K(z,w)h(z)^{-\varepsilon}$ is "worse" as its L^1 norm is algebraic rather than logarithmic.

A classical estimate

Theorem (Forelli-Rudin 80's)

For $\beta > -2$ *and any* $\varepsilon > 0$, *the following holds:*

$$\int_{\mathbb{D}} \frac{(1-w)^{-\varepsilon}}{|1-z\bar{w}|^2}|w|^{\beta}\, dV(w) \lesssim (1-|z|^2)^{-\varepsilon}.$$

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Hence we may apply Schur's lemma with $h(z) = 1 - |z|^2$.

The proof of the preceding estimate is well-known and uses non-trivial asymptotic analysis; an alternate proof makes use of a decomposition and residue calculus.

Essentially, we decompose into cases where $|z| < 1/2$ and $|z| \geq 1/2$ to handle the singularity in the denominator.

Case 1: $|z| < 1/2$, so that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\bar{w}|^2} |w|^{\beta} \, dV(w) \approx \int_{\mathbb{D}} (1-|w|^2)^{-\varepsilon} |w|^{\beta} \, dV(w) < \infty,$$

for $\beta > -2$ and $\varepsilon \in (0, 1)$.

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Case 2: $|z| \geq 1/2$ and further subdivide into $|w| < \frac{1}{2|z|}$ and $|w| \geq \frac{1}{2|z|}$:

If $|w| < \frac{1}{2|z|}$, then $|1 - z\bar{w}| \geq 1/2$ and we proceed as above.

If $|w| \geq \frac{1}{2|z|}$, then

$$\int_{|w|\geq \frac{1}{2|z|}} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\bar{w}|^2}\, dV(w)$$
$$= \int_{\frac{1}{2|z|}}^{1} r(1-r^2)^{-\varepsilon} \left(\int_0^{2\pi} \frac{d\theta}{1-2r|z|\cos\theta + r^2|z|^2} d\theta \right)\, dr.$$

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Using residue calculus, the inner integral is comparable to $(1-r^2|z|^2)^{-1}$, and

$$\int_{\frac{1}{2|z|}}^{1} \frac{r(1-r^2)^{-\varepsilon}}{1-r^2|z|^2} \lesssim (1-|z|^2)^{\varepsilon},$$

using a trivial overestimate.

Domains of lower regularity classes

If the boundary of Ω is non-smooth, the range of p for which the Bergman projection is L^p bounded on is potentially smaller than $(1, \infty)$. There are also function spaces that are norm-bounded only for the trivial exponent $p = 2$. Domains and function spaces that have been studied include

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The integral operator $B_{\mathbb{D}}$ is bounded on $L^p(\mathbb{D})$ for all $p \in (1, \infty)$.

The projection on the disk

On $\mathbb{D} = \{|z| < 1\}$,

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- The operator is unbounded on $L^1(\mathbb{D})$ (logarithmically)
- A Caldéron-Zygmund decomposition shows that B is of weak-type $(1, 1)$.

Historically, much work has been done in the case that our domain is smooth or well-behaved (unit ball, unit disk, polydisk $\mathbb{D}^n$, etc.)

Weak-type regularity

A linear operator T acting on $L^p(X)$ is said to be of weak-type (p, p) if

$$|\{x \in X : |Tf(x)| > \lambda\}| \leq c \frac{\|f\|_p^p}{\lambda^p},$$

with $c > 0$ independent of f.

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If T is bounded on L^p, it is of weak-type (p, p) by Chebychev's inequality. But weak-type regularity is useful because of Marcinkiewicz interpolation:

Theorem (Real interpolation)

If T is of weak-type (a, a) and of weak-type (b, b), then it is bounded on $L^p(X)$ for all $a < p < b$.

Weak-type regularity of other singular integral operators

Consider the Hardy-Littlewood maximal function on $\mathbb{R}$:

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

and the Hilbert transform:

$$Hf(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

- Both M and H are bounded in $L^p(\mathbb{R})$ for all $p \in (1, \infty)$.
- Both are unbounded on L^1, but they are of weak-type $(1, 1)$.

If given an operator that is known to be bounded on $L^p(X)$ for some $a < p < b$, analysts generally expect it to be of weak-type (a, a) and (b, b).

Generalized Hartogs triangles

The generalized Hartogs triangle, $\mathbb{H}_\gamma$, is given by

$$\mathbb{H}_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$$

for $\gamma > 0$.

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Theorem (Edholm-McNeal '16)

The Bergman projection on $\mathbb{H}_\gamma$ (as above) is bounded on $L^p(\mathbb{H}_\gamma)$ only for $p = 2$ if $\gamma \notin \mathbb{Q}^+$. If $\gamma = m/n \in \mathbb{Q}^+$, then the Bergman projection is bounded on $L^p(\mathbb{H}_{m/n})$ if and only if

$$p \in \left(\frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n-1} \right).$$

Our result in $\mathbb{C}^2$

As Huo-Wick did for the classical triangle, we proved that the Bergman projection on the rational Hartogs triangle $\mathbb{H}_{m/n}$ satisfies a weak-type estimate only at the upper-endpoint of L^p boundedness:

Theorem (C.-Koenig '22)

The Bergman projection on $\mathbb{H}_{m/n}$ is of weak-type $\left(\frac{2m+2n}{m+n-1}, \frac{2m+2n}{m+n-1}\right)$, but is not of weak-type $\left(\frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n+1}\right)$.

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Question

Does this phenomenon extend to higher dimsensions?

Weak-type regularity on triangles in $\mathbb{C}^3$

Consider the natural analogue to the rational Hartogs triangle in $\mathbb{C}^3$:

$$\mathbb{H}_{\mathbf{p}} = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^{p_1} < |z_2|^{p_2} < |z_3|^{p_3} < 1\right\},$$

where $\mathbf{p} \in \mathbb{N}^3$ and $\gcd(p_1, p_2, p_3) = 1$.

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where $\mathbf{p} \in \mathbb{N}^3$ and $\gcd(p_1, p_2, p_3) = 1$. Park (2019) computed an explicit form of the Bergman kernel for $\mathbb{H}_{\mathbf{p}}$ and S. Zhang (2021) showed that B is bounded on $L^p(\mathbb{H}_{\mathbf{p}})$ if and only if $p \in \left(\frac{2L}{L+D}, \frac{2L}{L-D}\right)$, where $L = p_1p_2 + p_1p_3 + p_2p_3$ and $D = \gcd(p_1p_2, p_1p_3, p_2p_3)$.

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Indeed, we observed the same one-sided weak-type phenomenon in $\mathbb{C}^3$:

Theorem (C.-Koenig 2023)

The Bergman projection on $\mathbb{H}_{\mathbf{p}}$ is not of weak-type $\left(\frac{2L}{L+D}, \frac{2L}{L+D}\right)$ but is of weak-type $\left(\frac{2L}{L-D}, \frac{2L}{L-D}\right)$.

Proof sketch of positive result

1. Obtain pointwise estimate on Bergman kernel using asymptotics
2. Transform integral for weak-type estimate on $\mathbb{H}_{\mathbf{p}}$ to integral on $\mathbb{D}^3$ using proper holomorphic maps.
3. Decompose integral on $\mathbb{D}^3$ into 8 cases, according to whether $|z_1|, |z_2|, |z_3|$ is close to zero or bounded away from zero.

Failure of weak-type regularity at lower endpoint $p = \frac{2L}{L+D}$

We must exhibit a counter-example $f_\lambda \in L^p(\mathbb{H}_\mathbf{p})$ and constants c_λ (with $c_\lambda \to \infty$ as $\lambda \to \infty$) such that

$$|\{(z_1, z_2, z_3) \in \mathbb{H}_\mathbf{p} : |Bf(z_1, z_2, z_3)| > \lambda\}| > c_\lambda \frac{\|f\|^p_{L^p(\mathbb{H}_\mathbf{p})}}{\lambda^p}.$$

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Namely,

$$f_\lambda(z_1, z_2, z_3) = f(z_1, z_2, z_3) = \bar{z}_1^{-k}|z_1|^k \bar{z}_2^{-\ell}|z_2|^\ell \bar{z}_3^{a_3}|z_3|^{b_3},$$

where $k, \ell \in \mathbb{N} \cup \{0\}$, $a_3 \in \mathbb{N}$ and $b_3 = b_3(\lambda) \in \mathbb{R}^+$.

(We define f to be zero if $z_1 = 0$ or $z_2 = 0$).

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(We define f to be zero if $z_1 = 0$ or $z_2 = 0$).

This function f is loosely based on the two-dimensional examples of Huo-Wick 2020 and C.-Koenig 2022.

Finding the L^p norm and Bergman projection for f

For such $f(z_1, z_2, z_3) = \bar{z}_1^{-k}|z_1|^k \bar{z}_2^{-\ell}|z_2|^\ell \bar{z}_3^{a_3}|z_3|^{b_3}$,

$$\|f\|^p_{L^p(\mathbb{H}_\mathbf{p})} = \frac{2\pi^3}{\left(\frac{p_1}{p_2}+1\right)\left(p(a_3+b_3)+\frac{2p_3}{p_1}+\frac{2p_3}{p_2}+2\right)}$$

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Finding a closed form for the Bergman kernel on $\mathbb{H}_\mathbf{p}$ is difficult (see, for example, J.-D. Park 2019).

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In our case, suitably normalized sums of three-variable monominals can be taken as the orthonormal basis:

$$\frac{w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3}}{\| w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3} \|_{L^2(\mathbb{H}_{\mathbf{p}})}^2}, \tag{1}$$

provided that the monomial $w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3}$ is square integrable near the origin.

Calculating the Bergman projection $Bf(z_1, z_2, z_3)$

For our f,

$$
\begin{aligned}
&Bf(z_1, z_2, z_3) \\
&= \int_{\mathbb{H}_{\mathbf{p}}} \sum_{\alpha \in \mathcal{A}} \frac{(z_1 \bar{w}_1)^{\alpha_1} (z_2 \bar{w}_2)^{\alpha_2} (z_3 \bar{w}_3)^{\alpha_3}}{\| w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3} \|^2_{L^2(\mathbb{H}_{\mathbf{p}})}} \bar{w}_1^{-k} |w_1|^k \bar{w}_2^{-\ell} |w_2|^{\ell} \bar{w}_3^{a_3} |w_3|^{b_3} \\
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The set $\mathcal{A}$ of allowable indices consists of the triples $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ satisfying the following three conditions:

Calculating the Bergman projection $Bf(z_1, z_2, z_3)$

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$$\alpha_1 \geq 0,$$
$$\alpha_2 + \frac{p_2}{p_1}(\alpha_1 + 1) > -1,$$
$$\alpha_3 + \frac{p_3}{p_2}(\alpha_2 + \frac{p_2}{p_1}(\alpha_1 + 1) + 1) > -1.$$

Taking advantage of orthogonality

By orthogonality,

$$= \int_{\mathbb{H}_{\mathbf{p}}} \sum_{\alpha \in \mathcal{A}} \frac{(z_1 \bar{w}_1)^{\alpha_1} (z_2 \bar{w}_2)^{\alpha_2} (z_3 \bar{w}_3)^{\alpha_3}}{\| w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3} \|^2_{L^2(\mathbb{H}_{\mathbf{p}})}} \bar{w}_1^{-k} |w_1|^k \bar{w}_2^{-\ell} |w_2|^\ell \bar{w}_3^{a_3} |w_3|^{b_3} \, dV(w_1, w_2, w_3)$$

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Since $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, it must be the case that k, ℓ, and a_3 are integers.

Selecting parameters

If $p_1 = p_2 = 1$, we take $\ell = k = 0$ so

$$f(z_1, z_2, z_3) = \bar{z}_3^{a_3} |z_3|^{b_3}$$

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$$\mathbb{H}_{(1,1,p_3)} = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| < |z_2| < |z_3|^{p_3}\right\} \cong \mathbb{D} \times \mathbb{D}^* \times \mathbb{D}^*.$$

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Otherwise, if $p_1 p_2 > 1$, we choose positive integers ℓ and k such that

$$\ell p_1 p_3 + k p_2 p_3 + L - D \equiv 0 \bmod (p_1 p_2),$$

or equivalently,

$$(\ell + 1)(p_1 p_3) + (k + 1)(p_2 p_3) \equiv D \bmod (p_1 p_2).$$

Bézout's identity and allowable indices

By Bézout's identity, there exist $x_1, x_2, x_3 \in \mathbb{Z}$ such that

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Adding a multiple of $p_1 p_2$ to x_1 and x_2 if necessary, we may assume $x_1, x_2 \in \mathbb{N}$ and then take $\ell + 1 = x_1$ and $k + 1 = x_2$. Thereby ensuring that $k, \ell \in \mathbb{Z}$.

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Note that $\alpha_1 = k \geq 0$ and $\ell + \frac{p_2}{p_1}(k+1) > -1$ since $\ell \geq 0$. Finally, let

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Therefore, the triple $(k, \ell, -a_3) \in \mathcal{A} \subset \mathbb{Z}^3$.

Consequently,

$$Bf(z_1, z_2, z_3) = \frac{z_1^k z_2^\ell z_3^{-a_3}}{\|w_1^k w_2^\ell w_3^{-a_3}\|^2_{L^2(\mathbb{H}_{\mathbf{p}})}} \int_{\mathbb{H}_{\mathbf{p}}} |w_1|^k |w_2|^\ell |w_3|^{b_3} \, dV(w_1, w_2, w_3)$$
$$= \frac{8\pi^3 z_1^k z_2^\ell z_3^{-a_3}}{c'(k+2)\left(\ell + \frac{p_2}{p_1}(k+2) + 2\right)\left(b_3 + \frac{p_3}{p_2}\left(\ell + \frac{p_2}{p_1}(k+2) + 2\right) + 2\right)}.$$

Selecting $b_3 \in \mathbb{R}$

Working with the quantity

$$\frac{8\pi^3}{c'(k+2)\left(\ell+\frac{p_2}{p_1}(k+2)+2\right)\left(b_3+\frac{p_3}{p_2}\left(\ell+\frac{p_2}{p_1}(k+2)+2\right)+2\right)},$$

set $\gamma = \ell + \frac{p_2}{p_1}(k+2)+2$ and $b_3 = \lambda^{-\delta} - \frac{p_3}{p_2}\gamma - 2$, where $\delta > 0$ is to be determined.

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set $\gamma = \ell + \frac{p_2}{p_1}(k+2)+2$ and $b_3 = \lambda^{-\delta} - \frac{p_3}{p_2}\gamma - 2$, where $\delta > 0$ is to be determined.

With this choice,

$$Bf(z_1, z_2, z_3) = c z_1^k z_2^\ell z_3^{-a_3} \lambda^\delta,$$

and

$$\|f\|^p_{L^p(\mathbb{H}_\mathbf{p})} = \frac{2\pi^3}{p\left(\frac{p_2}{p_1}+1\right)} \cdot \lambda^\delta.$$

with $c > 0$ independent of λ.

Failure of weak-type estimate

Define

$$\widetilde{\mathbb{H}} := \left\{ (z_1, z_2, z_3) \in \mathbb{H}_{\mathbf{p}} : \left(\frac{1}{2}|z_3| \right)^{p_3} < |z_1|^{p_1} < |z_2|^{p_2} < |z_3|^{p_3} < 1 \right\}.$$

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For $\lambda > 0$,

$$\begin{aligned}
&|\{(z_1, z_2, z_3) \in \mathbb{H}_{\mathbf{p}} : |Bf(z_1, z_2, z_3)| > \lambda\}| \\
&\geq \left| \left\{ (z_1, z_2, z_3) \in \widetilde{\mathbb{H}} : c|z_1^k z_2^\ell z_3^{-a_3}|\lambda^\delta > \lambda \right\} \right| \\
&\geq \left| \left\{ (z_1, z_2, z_3) \in \widetilde{\mathbb{H}} : c'|z_3|^{\frac{p_3}{p_1}k + \frac{p_3}{p_2}\ell - a_3} \lambda^\delta > \lambda \right\} \right| \\
&= \left| \left\{ (z_1, z_2, z_3) \in \widetilde{\mathbb{H}} : |z_3| < \left(c'\lambda^{\delta - 1} \right)^{\frac{1}{a_3 - \frac{p_3}{p_2}\ell - \frac{p_3}{p_1}k}} \right\} \right| \\
&= \left| \left\{ (z_1, z_2, z_3) \in \widetilde{\mathbb{H}} : |z_3| < c'' \left(\lambda^{\delta - 1} \right)^{\frac{p_1 p_2}{L - D}} \right\} \right|,
\end{aligned}$$

In the region

$$\left|\left\{(z_1, z_2, z_3) \in \widetilde{\mathbb{H}} : |z_3| < c'' \left(\lambda^{\delta-1}\right)^{\frac{p_1 p_2}{L-D}}\right\}\right|$$

note that $|z_3| < 1$ for λ sufficiently large provided that $\delta \in (0, 1)$.

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note that $|z_3| < 1$ for λ sufficiently large provided that $\delta \in (0, 1)$. For such λ,

$$(2) = \pi^2 \int_{\left\{|z_3| < c''\left(\lambda^{\delta-1}\right)^{\frac{p_1 p_2}{L-d}}\right\}} \eta |z_3|^{\frac{2p_3}{p_1} + \frac{2p_3}{p_2}} \, dV(z_3),$$

where

$$\eta = \frac{1 - 2^{-\left(\frac{2p_3}{p_1} + \frac{2p_3}{p_2}\right)}}{\frac{p_2}{p_1} + 1} - 2^{-\frac{2p_3}{p_1}} (1 - 2^{\frac{-2p_3}{p_2}}) > 0$$

Parameter blowup

Finally,

$$\pi^2 \int_{\left\{|z_3|<c''\left(\lambda^{\delta-1}\right)^{\frac{p_1 p_2}{L-d}}\right\}} \eta |z_3|^{\frac{2p_3}{p_1}+\frac{2p_3}{p_2}} \; dV(z_3)$$

$$= \frac{\pi^3 \eta}{\frac{p_3}{p_1}+\frac{p_3}{p_2}+1}\left(c''\lambda^{(\delta-1)\frac{p_1 p_2}{L-D}}\right)^{\frac{2p_3}{p_1}+\frac{2p_3}{p_2}+2}$$

$$\approx \lambda^{(\delta-1)p'} \approx \frac{\|f\|^p_{L^p(\mathbb{H}_{\mathbf{p}})}}{\lambda^p}\lambda^{(\delta-1)p'+p-\delta}.$$

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$$= \frac{\pi^3 \eta}{\frac{p_3}{p_1}+\frac{p_3}{p_2}+1}\left(c''\lambda^{(\delta-1)\frac{p_1 p_2}{L-D}}\right)^{\frac{2p_3}{p_1}+\frac{2p_3}{p_2}+2}$$

$$\approx \lambda^{(\delta-1)p'} \approx \frac{\|f\|^p_{L^p(\mathbb{H}_{\mathbf{p}})}}{\lambda^p}\lambda^{(\delta-1)p'+p-\delta}.$$

Note that $1<p<p'$ so by choosing $\delta \in \left(\frac{p'-p}{p'-1}, 1\right)$, the factor $\lambda^{(\delta-1)p'+p-\delta}$ blows up as $\lambda \to \infty$. (Recall $\|f\|^p_{L^p(\mathbb{H}_{\mathbf{p}})} \approx \lambda^\delta$).
Therefore, the weak-type estimate fails.

Thank you!

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