Complex Linear Algebra without Complex Numbers

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Can one do complex linear algebra without using complex numbers?

Yes …

You just need a real vector space V together with a real linear operator $J : V \to V$ satisfying $J^2 = -I$.
Even for $V = \mathbb{R}^2$, there are many such J:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 47 & -34 \\ 65 & -47 \end{bmatrix}, \begin{bmatrix} -5 & 26 \\ -1 & 5 \end{bmatrix}, \ldots$$

You *could* then define a complex vector space by using this formula for scalar multiplication: $(a+ib)\bullet v = a\bullet v + b\bullet J(v)$, but *you don't have to*!

But … Why?

In linear algebra over $\mathbb{R}$, using complex structure operators J is more convenient than complex scalars when you need to keep track of MORE THAN ONE complex structure. This happens in several contexts:

1. One vector space V can have two complex structures: J_1 and J_2.
   a. Maybe they anticommute: $J_1 J_2 = -J_2 J_1$. Then V has a quaternionic structure!
   b. Maybe they commute: $J_1 J_2 = J_2 J_1$. Then (by the exercise) $J_1 J_2$ is an involution and V is a direct sum of its $+1$, -1 eigenspaces. For a vector v in the (-1)-subspace, $J_1 J_2(v) = -v \Leftrightarrow J_2(v) = J_1(v)$, the two complex structures are equal, and similarly on the ($+1$)-subspace, they are opposite: $J_2(v) = -J_1(v)$.

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Here's a linear algebra problem you can do in your head!

$$\text{Assume: } J_1^{\,2} = J_2^{\,2} = -I$$

$$\text{Prove: } J_1 J_2 = J_2 J_1 \Leftrightarrow (J_1 J_2)^2 = I$$

More information about complex structures is available on my web site:
`users.pfw.edu/CoffmanA/`

- For more about linear algebra, complex structures, and the trace, see my *Notes on Abstract Linear Algebra*
- For my research on complex structures in geometric analysis, see my joint papers with **Yifei Pan** or **Yuan Zhang**

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2. Two vector spaces can each have a complex structure: J_1 on V_1 and J_2 on V_2.
   a. The set of real linear maps from V_1 to V_2 is also a real vector space, $\mathrm{Hom}(V_1,V_2)$, and it has two commuting complex structures: $A \mapsto A\circ J_1$ and $A \mapsto J_2\circ A$. This is a special case of 1.b.: the subspace where $A\circ J_1 = J_2\circ A$ is the set of maps that are "complex linear" with respect to the complex structures on the domain and target.
   b. The real tensor product $V_1 \otimes V_2$ also admits commuting complex structures, $[J_1 \otimes I]$ and $[I \otimes J_2]$. The subspace where they agree is the complex tensor product.

3. A complex structure may depend on variables like position or time. In differential geometry, at points x on a manifold, each tangent space T_x may have a complex structure J_x. Some of my recent research in geometric analysis considers complex structures that depend continuously, but not smoothly, on x.

How do I find the Trace of a complex linear map $A : V \to V$?

You don't.

Even though the scalar-valued Trace is important in both algebra and differential geometry, it turns out not to be a natural concept in this without-complex-numbers framework. The right way to do things is suggested by Category Theory, where the **Generalized Trace** of a map $A : V\otimes U \to V\otimes W$ is another map $\mathrm{Tr}(A) : U \to W$. All you need is:

- an **Evaluation** map $\varepsilon : \mathrm{Hom}(V,W) \otimes V \to W$
- & a **Co-Evaluation** $\eta : U \to V \otimes \mathrm{Hom}(V,U)$

related by certain identities. $\mathrm{Tr}(A)$ is the composite

$$U \to V\otimes\mathrm{Hom}(V,U) \to \mathrm{Hom}(V,V\otimes U) \to \mathrm{Hom}(V,V\otimes W) \to \mathrm{Hom}(V,W)\otimes V \to W$$

where the second and fourth arrows are natural isomorphisms and the middle arrow is $F \mapsto A \circ F$. This construction can be adapted to maps that are complex linear with respect to complex structure operators on V, U, and W.