Zeros of Ultraspherical Polynomials

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Special case $\alpha = \beta = \lambda - \frac{1}{2}$ of Jacobi polynomials $P_n^{(\alpha,\beta)}$

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The zeros of $C_{n-1}^{(\lambda)}$ and $C_n^{(\lambda)}$ are interlacing:

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D-Duren 2000. A specific number of zeros of $C_n^{(\lambda)}$ collide at the endpoints 1 and -1 of the interval of orthogonality each time λ decreases through the next successive negative half-integer. The location and kinematics of the zeros at each stage of the process are known.

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Quasi-orthogonality:Riesz (Moment Problem), Fejer, Shohat, Chihara...

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D-Muldoon 2015 For $\lambda < 1 - n$, zeros of $\tilde{C}_{n-1}^{(\lambda)}$ interlace zeros of $\tilde{C}_n^{(\lambda)}$

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Let $\tilde{x}_{n,k}(\lambda), k = 1, ..., n$ be the zeros of $\tilde{C}_n^{(\lambda)}$

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Two real symmetric zeros of ultra have modulus > 1

If $1 - n < \lambda < 2 - n$ two zeros of $\tilde{C}_n^{(\lambda)}$ are pure imaginary

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Follows from the bounds derived in D-Muldoon 2016. The extreme zeros of $C_n^{(\lambda)}(x)$ approach ± 1 as n approaches ∞.

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Weight function pseudo-ultras $\lambda < -n$ is $(1+x^2)^{\lambda-1/2}$

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Thank you