Strong Szegő Limit Theorems for Bordered and Framed Toeplitz Determinants

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Based on :

- *Asymptotics of bordered Toeplitz determinants and next-to-diagonal Ising correlations.* Journal of Statistical Physics (2022). Estelle Basor, Torsten Ehrhardt, R G, Alexander Its, and Yuqi Li.
- *Strong Szegő limit theorems for multi-bordered, framed, and multi-framed Toeplitz determinants.* arXiv:2309.14695 (2023). R G.
- *Deformed Toeplitz and Hankel determinants.* In preparation. R G, and Karl Liechty.

The $N \times N$ **Toeplitz** matrix associated to the symbol ϕ is defined as

$$T_N[\phi] = \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-N+1} \\ \phi_1 & \phi_0 & \cdots & \phi_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N-1} & \phi_{N-2} & \cdots & \phi_0 \end{pmatrix},$$

where ϕ_k's are the Fourier coefficients of ϕ

$$\phi_k = \int_{\mathbb{T}} z^{-k} \phi(z) \frac{\mathrm{d}z}{2\pi \mathrm{i} z}.$$

Let

$$D_N[\phi] := \det T_N[\phi].$$

The large-N asymptotics of the Toeplitz determinants are well known and given by the **Szegö-Widom theorem** by

$$D_N[\phi] \sim G[\phi]^N E[\phi],$$

$$G[\phi] = \exp\left([\log \phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n \geq 1} n [\log \phi]_n [\log \phi]_{-n}\right).$$

Let Q_n and $\widehat{Q}_n$ be respectively defined by

$$Q_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix},$$

and

$$\widehat{Q}_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+1} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+2} & z \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_n & \phi_{n-1} & \cdots & \phi_1 & z^n \end{pmatrix},$$

$$Q_n(z) = \kappa_n z^n + \sum_{\ell=0}^{n-1} \kappa_\ell^{(n)} z^\ell, \qquad \text{and} \qquad \widehat{Q}_n(z) = \kappa_n z^n + \sum_{\ell=0}^{n-1} \widehat{\kappa}_\ell^{(n)} z^\ell,$$

where

$$\kappa_n = \sqrt{\frac{D_n[\phi]}{D_{n+1}[\phi]}}.$$

Let Q_n and $\widehat{Q}_n$ be respectively defined by

$$Q_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix},$$

and

$$\widehat{Q}_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+1} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+2} & z \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_n & \phi_{n-1} & \cdots & \phi_1 & z^n \end{pmatrix},$$

One can readily observe that $\{Q_n\}_{n=0}^{\infty}$ and $\{\widehat{Q}_n\}_{n=0}^{\infty}$ form the **bi-orthogonal system of polynomials** on the unit circle with respect to the weight ϕ :

$$\int_{\mathbb{T}} Q_n(z)\widehat{Q}_n(z^{-1})\phi(z)\frac{\mathrm{d}z}{2\pi \mathrm{i} z} = \delta_{nk}, \qquad n, k = 0, 1, 2, \cdots .$$

It is due to J.Baik, P.Deift and K.Johansson that the following matrix-valued function constructed out of the polynomials Q_n and $\widehat{Q}_n$

$$X(z;n) := \begin{pmatrix} \kappa_n^{-1} Q_n(z) & \kappa_n^{-1} \int_{\mathbb{T}} \frac{Q_n(\zeta)}{(\zeta - z)} \frac{\phi(\zeta)\mathrm{d}\zeta}{2\pi \mathrm{i} \zeta^n} \\ -\kappa_{n-1} z^{n-1} \widehat{Q}_{n-1}(z^{-1}) & -\kappa_{n-1} \int_{\mathbb{T}} \frac{\widehat{Q}_{n-1}(\zeta^{-1})}{(\zeta - z)} \frac{\phi(\zeta)\mathrm{d}\zeta}{2\pi \mathrm{i} \zeta} \end{pmatrix},$$

satisfies the following **Riemann-Hilbert problem** for the BOPUC, which in the subsequent parts of this text will occasionally be referred to as the X-RHP:

- **RH-X1** $X : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-X2** The limits of $X(\zeta)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, and are denoted $X_\pm(z)$ respectively and are related by

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T},$$

- **RH-X3** As $z \to \infty$

$$X(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Let us first recall the two-dimensional **Ising model**, solved by Onsager. In this model a $2\mathcal{M} \times 2\mathcal{N}$ rectangular lattice is considered with an associated spin variable σ_{jk} taking the values 1 and -1 at each vertex (j, k), $-\mathcal{M} \leq j \leq \mathcal{M} - 1$, $-\mathcal{N} \leq k \leq \mathcal{N} - 1$. There are $2^{4\mathcal{M}\mathcal{N}}$ possible spin configurations $\{\sigma\}$ of the lattice (a configuration corresponds to values of all σ_{jk} fixed). By $\mathcal{J}_h$ and $\mathcal{J}_v$ we respectively denote the horizontal and vertical nearest neighbor coupling constants and with each configuration we associate its **nearest-neighbor coupling energy** given by

$$E(\{\sigma\}) = -\sum_{j=-\mathcal{M}}^{\mathcal{M}-1} \sum_{k=-\mathcal{N}}^{\mathcal{N}-1} \left(\mathcal{J}_h \sigma_{j,k} \sigma_{j,k+1} + \mathcal{J}_v \sigma_{j,k} \sigma_{j+1,k}\right), \qquad \mathcal{J}_h, \mathcal{J}_v > 0.$$

The probability of a spin configuration $\{\sigma\}$ is given by

$$P_{\{\sigma\}} = \frac{1}{Z(T)} \exp\left(-\frac{E(\{\sigma\})}{k_B T}\right),$$

where k_B is the Boltzmann's constant and $Z(T)$ denotes the partition function and is naturally defined as

$$Z(T) = \sum_{\{\sigma\}} \exp\left(-\frac{E(\{\sigma\})}{k_B T}\right).$$

The spin-spin correlation function between the vertices (m', n') and (m, n) is defined as the following *thermodynamic limit*

$$\langle \sigma_{m',n'} \sigma_{m,n} \rangle = \lim_{\mathcal{M},\mathcal{N}\to\infty} \frac{1}{Z(T)} \sum_{\{\sigma\}} \sigma_{m',n'} \sigma_{m,n} \exp\left(-\frac{E(\{\sigma\})}{k_B T} \right).$$

The quantity $\lim_{m^2+n^2\to\infty} \langle \sigma_{0,0} \sigma_{m,n} \rangle$ is referred to as the **long-range order** in the lattice at a temperature T. Indeed, the spontaneous magnetization M is defined as square of the large-n limit of *diagonal* correlations

$$M := \sqrt{\lim_{n\to\infty} \langle \sigma_{0,0} \sigma_{n,n} \rangle}.$$

Let us introduce the notations,

$$S_h = \sinh\left(\frac{2\mathcal{J}_h}{k_B T} \right), \quad S_v = \sinh\left(\frac{2\mathcal{J}_v}{k_B T} \right),$$

$$C_h = \cosh\left(\frac{2\mathcal{J}_h}{k_B T} \right), \quad C_v = \cosh\left(\frac{2\mathcal{J}_v}{k_B T} \right),$$

and

$$k = S_h S_v.$$

It is famously known that, unlike the one-dimensional case, the two-dimensional Ising model exhibits a phase transition in the spontaneous magnetization at some temperature T_c, characterized by

$$k = 1.$$

In this talk I will focus on

$$k > 1,$$

which corresponds to the low temperature regime $T < T_c$.

For the diagonal correlations $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ and the horizontal correlations $\langle \sigma_{0,0}\sigma_{0,N} \rangle$, one has Toeplitz determinant representations. Indeed, for the diagonal correlations we have

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = D_N[\widehat{\phi}\,], \qquad \widehat{\phi}(z) = \sqrt{\frac{1 - k^{-1}z^{-1}}{1 - k^{-1}z}},$$

and for the horizontal correlations

$$\langle \sigma_{0,0}\sigma_{0,N} \rangle = D_N[\widehat{\eta}\,], \qquad \widehat{\eta}(z) = \sqrt{\frac{(1-\alpha_1 z)(1-\alpha_2 z^{-1})}{(1-\alpha_1 z^{-1})(1-\alpha_2 z)}},$$

where α_1 and α_2 are given by

$$\alpha_1 = \frac{z_h(1-z_v)}{1+z_v}, \quad \alpha_2 = \frac{1-z_v}{z_h(1+z_v)}, \qquad z_{h,v} = \tanh\frac{\mathcal{J}_{h,v}}{k_B T}.$$

In the low temperature regime, the symbols $\widehat{\phi}$ and $\widehat{\eta}$ enjoy the regularity properties required by the strong Szegő limit theorem and the diagonal and horizontal long-range orders

$$M_D := \sqrt{\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{N,N} \rangle} \quad \text{and} \quad M_H := \sqrt{\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{0,N} \rangle},$$

both evaluate to

$$(1-k^{-2})^{1/8}.$$

In an interesting development, It was shown by Au-Yang and Perk in 1987 that the **next-to-diagonal** two point correlation function is given by the following bordered Toeplitz determinant,

$$\langle \sigma_{0,0}\sigma_{N-1,N} \rangle = D_N^B[\widehat{\phi}, \widehat{\psi}],$$

where $\widehat{\phi}$ is the symbol for diagonal correlations, and

$$\widehat{\psi}(z) = \frac{C_v z \widehat{\phi}(z) + C_h}{S_v(z-c_*)}, \qquad \text{with} \qquad c_* = -\frac{S_h}{S_v}.$$

In an interesting development, It was shown by Au-Yang and Perk in 1987 that the **next-to-diagonal** two point correlation function is given by the following bordered Toeplitz determinant,

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The **bordered Toeplitz determinant**, $D_N^B[\phi;\psi]$, is defined as

$$D_N^B[\phi;\psi] := \det\begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{N-2} & \psi_{N-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{N-3} & \psi_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2-N} & \phi_{3-N} & \cdots & \phi_0 & \psi_1 \\ \phi_{1-N} & \phi_{2-N} & \cdots & \phi_{-1} & \psi_0 \end{pmatrix}, \qquad N > 1.$$

Theorem 1. Let $D_N^B[\phi;\psi]$ be the bordered Toeplitz determinant with $\psi = q_1\phi + q_2$, where

$$q_1(z) = a_0 + a_1 z + \frac{b_0}{z} + \sum_{j=1}^{m} \frac{b_j z}{z - c_j}, \quad \text{and} \quad q_2(z) = \hat{a}_0 + \hat{a}_1 z + \frac{\hat{b}_0}{z} + \sum_{j=1}^{m} \frac{\hat{b}_j}{z - c_j},$$

and ϕ of Szegő type. Then, as $N \to \infty$

$$D_N^B[\phi, \psi] = G[\phi]^N E[\phi] \left(F[\phi;\psi] + \mathcal{O}(\rho^{-N}) \right),$$

where

$$G[\phi] = \exp\left([\log\phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n\geq 1} n[\log\phi]_n [\log\phi]_{-n} \right),$$

$$F[\phi;\psi] = a_0 + b_0[\log\phi]_1 + \sum_{\substack{j=1 \\ 0<|c_j|<1}}^{m} b_j \frac{\alpha(c_j)}{\alpha(0)} + \frac{1}{\alpha(0)} \left(\hat{a}_0 - \hat{a}_1[\log\phi]_{-1} - \sum_{\substack{j=1 \\ |c_j|>1}}^{m} \frac{\hat{b}_j}{c_j}\alpha(c_j) \right),$$

$$\alpha(z) := \exp\left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau \right],$$

Theorem 2. Let $\langle \sigma_{0,0}\sigma_{N-1,N}\rangle$ be the next-to-diagonal two point correlation function in the Ising model. Then, in the low-temperature regime, the long-range order in the next-to-diagonal direction for the anisotropic square lattice Ising model is the same as of the diagonal and horizontal ones, i.e. is described as follows

$$\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{N-1,N}\rangle = (1-k^{-2})^{1/4}.$$

Theorem 3. The next-to-diagonal two point correlation function has, in the low-temperature regime $k > 1$, the $N \to \infty$ asymptotics

$$\langle \sigma_{0,0}\sigma_{N-1,N}\rangle = (1-k^{-2})^{1/4}\left(1+\frac{1}{2\pi(1-k^{-2})}\left(\frac{1}{C_v^2}+\frac{1}{k^2-1}\right)N^{-2}k^{-2N}\left(1+O(N^{-1})\right)\right)$$

For comparison, asymptotics of the diagonal correlation function is given by

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-k^{-2})^{1/4}\left(1+\frac{1}{2\pi(1-k^{-2})^2k^2}N^{-2}k^{-2N}\left(1+O(N^{-1})\right)\right),$$

as $N \to \infty$.

Theorem 4. Suppose that $\psi(z)$ admits an analytic continuation in some neighborhood of the unit circle and let ϕ be of Szegő type. Then

$$D_N^B[\phi,\psi] = G[\phi]^N E[\phi]\left(F[\phi;\psi] + \mathcal{O}(e^{-\mathfrak{c}N})\right),$$

where

$$G[\phi] = \exp\left([\log\phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n\geq 1} n[\log\phi]_n[\log\phi]_{-n}\right),$$

and $F[\phi;\psi]$ is given by

$$F[\phi;\psi] = \frac{[\alpha_-\psi]_0}{\alpha(0)} \equiv \frac{1}{\alpha(0)}\int_{\mathbb{T}} \alpha_-(w)\psi(w)\frac{\mathrm{d}w}{2\pi \mathrm{i} w},$$

and $\mathfrak{c}$ is some positive constant.

The bordered Toeplitz determinants $D^B_{n+1}[\phi, \frac{1}{z-c}]$, and $D^B_{n+1}[\phi, \frac{\phi}{z-c}]$ are encoded into X-RHP data described by

$$D^B_{n+1}[\phi, \frac{1}{z-c}] = \begin{cases} 0, & |c| < 1, \\ -c^{-n-1} D_n[\phi] X_{11}(c;n), & |c| > 1, \end{cases}$$

and

$$D^B_{n+1}[\phi, \frac{\phi}{z-c}] = -\frac{1}{c} D_{n+1}[\phi] + \frac{1}{c} D_n[\phi] X_{12}(c,n), \qquad c \neq 0,$$

$$D^B_{n+1}[\phi; z^{-\ell}\phi] = \frac{D_n[\phi]}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} X_{12}(z;n)\bigg|_{z=0},$$

$$D^B_{n+1}[\phi, z] = D_n[\phi] \lim_{z\to\infty} \left(\frac{X_{11}(z;n) - z^n}{z^{n-1}} \right) \equiv D_n[\phi] \frac{\kappa^{(n)}_{n-1}}{\kappa_n},$$

$$D^B_{n+1}[\phi, z^2] = D_n[\phi] \lim_{z\to\infty} \left(\frac{X_{11}(z;n) - z^n - \frac{\kappa^{(n)}_{n-1}}{\kappa_n} z^{n-1}}{z^{n-2}} \right) \equiv D_n[\phi] \frac{\kappa^{(n)}_{n-2}}{\kappa_n},$$

and so on. Note that

$$D^B_{n+1}[\phi, z^k] = 0, \qquad k \in \mathbb{Z} \setminus \{0, 1, \cdots n\}.$$

$$D_n^B[\phi;\boldsymbol{\psi}_m] := \det\begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{n-m-1} & \psi_{1,n-1} & \psi_{2,n-1} & \cdots & \psi_{m,n-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{n-m-2} & \psi_{1,n-2} & \psi_{2,n-2} & \cdots & \psi_{m,n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_{-n+1} & \phi_{-n+2} & \cdots & \phi_{-m} & \psi_{1,0} & \psi_{2,0} & \cdots & \psi_{m,0} \end{pmatrix},$$

For example, here you see the determinant $\mathscr{F}_n^{(3)}\left[\phi;\boldsymbol{\xi}_3,\boldsymbol{\psi}_3,\boldsymbol{\eta}_3,\boldsymbol{\gamma}_3;\boldsymbol{a}_{12}\right]$ with colored entries for easier interpretation:

$$\det\begin{pmatrix} a_9 & \xi_{3,n-3} & \xi_{3,n-4} & \xi_{3,n-5} & \xi_{3,n-6} & \cdots & \xi_{3,2} & \xi_{3,1} & \xi_{3,0} & a_{10} \\ \gamma_{3,n-3} & a_5 & \xi_{2,n-5} & \xi_{2,n-6} & \xi_{2,n-7} & \cdots & \xi_{2,1} & \xi_{2,0} & a_6 & \psi_{3,0} \\ \gamma_{3,n-4} & \gamma_{2,n-5} & a_1 & \xi_{1,n-7} & \xi_{1,n-8} & \cdots & \xi_{1,0} & a_2 & \psi_{2,0} & \psi_{3,1} \\ \gamma_{3,n-5} & \gamma_{2,n-6} & \gamma_{1,n-7} & \phi_0 & \phi_{-1} & \cdots & \phi_{-n+7} & \psi_{1,0} & \psi_{2,1} & \psi_{3,2} \\ \gamma_{3,n-6} & \gamma_{2,n-7} & \gamma_{1,n-8} & \phi_1 & \phi_0 & \cdots & \phi_{-n+8} & \psi_{1,1} & \psi_{2,2} & \psi_{3,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{3,2} & \gamma_{2,1} & \gamma_{1,0} & \phi_{n-7} & \phi_{n-8} & \cdots & \phi_0 & \psi_{1,n-7} & \psi_{2,n-6} & \psi_{3,n-5} \\ \gamma_{3,1} & \gamma_{2,0} & a_4 & \eta_{1,n-7} & \eta_{1,n-8} & \cdots & \eta_{1,0} & a_3 & \psi_{2,n-5} & \psi_{3,n-4} \\ \gamma_{3,0} & a_8 & \eta_{2,n-5} & \eta_{2,n-6} & \eta_{2,n-7} & \cdots & \eta_{2,1} & \eta_{2,0} & a_7 & \psi_{3,n-3} \\ a_{12} & \eta_{3,n-3} & \eta_{3,n-4} & \eta_{3,n-5} & \eta_{3,n-6} & \cdots & \eta_{3,2} & \eta_{3,1} & \eta_{3,0} & a_{11} \end{pmatrix}.$$

Theorem

For $\ell = 1, 2$, let $\psi_\ell(z) = q_1^{(\ell)}(z)\phi(z) + q_2^{(\ell)}(z)$ where

$$q_1^{(\ell)}(z) = a_0^{(\ell)} + a_1^{(\ell)} z + \frac{b_0^{(\ell)}}{z} + \sum_{j=1}^{m_\ell} \frac{b_j^{(\ell)} z}{z - c_j^{(\ell)}}, \quad \text{and} \quad q_2^{(\ell)}(z) = \hat{a}_0^{(\ell)} + \hat{a}_1^{(\ell)} z + \frac{\hat{b}_0^{(\ell)}}{z} + \sum_{j=1}^{m} \frac{\hat{b}_j^{(\ell)}}{z - c_j^{(\ell)}},$$

and suppose that ϕ is of Szegő-type. Then,

$$D_n^B[\phi; \boldsymbol{\psi}_2] \equiv D_n^B[\phi; \psi_1, \psi_2] = G^n[\phi]E[\phi]\left\{\mathcal{J}_1[\phi, \psi_1, \psi_2] + O(\rho^{-n})\right\},$$

$$\mathcal{J}_1[\phi, \psi_1, \psi_2] = \begin{vmatrix} F[\phi, \psi_2] & F[\phi, \psi_1] \\ H[\phi, \psi_2] & H[\phi, \psi_1] \end{vmatrix},$$

in which $F[\phi, \psi]$ is given by (9), and

$$H[\phi; \psi] = a_1 - \sum_{j=1}^{m} \frac{b_j}{c_j} + a_0[\log\phi]_1 + b_0[\log\phi]_2 + \frac{b_0}{2}[\log\phi]_1^2$$
$$+ \frac{1}{G[\phi]}\left(\hat{a}_1 - \sum_{\substack{j=1 \\ |c_j|>1}}^{m} \frac{\hat{b}_j}{c_j^2}\alpha(c_j) + \sum_{\substack{j=1 \\ 0<|c_j|<1}}^{m} \frac{b_j}{c_j}\alpha(c_j) \right).$$

Let

$$D_n^B[\phi;\boldsymbol{\psi}_2] \equiv \mathscr{D},$$

where

$$D_n^B[\phi;\boldsymbol{\psi}_2] = \det\begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{n-3} & \psi_{1,n-1} & \psi_{2,n-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{n-4} & \psi_{1,n-2} & \psi_{2,n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \phi_{-n+3} & \phi_{-n+4} & \cdots & \phi_0 & \psi_{1,2} & \psi_{2,2} \\ \phi_{-n+2} & \phi_{-n+3} & \cdots & \phi_{-1} & \psi_{1,1} & \psi_{2,1} \\ \phi_{-n+1} & \phi_{-n+2} & \cdots & \phi_{-2} & \psi_{1,0} & \psi_{2,0} \end{pmatrix}.$$

Consider the following Dodgson condensation identity:

$$\mathscr{D}\cdot\mathscr{D}\begin{Bmatrix} 0 & n-1 \\ n-2 & n-1 \end{Bmatrix} = \mathscr{D}\begin{Bmatrix} 0 \\ n-2 \end{Bmatrix}\cdot\mathscr{D}\begin{Bmatrix} n-1 \\ n-1 \end{Bmatrix} - \mathscr{D}\begin{Bmatrix} 0 \\ n-1 \end{Bmatrix}\cdot\mathscr{D}\begin{Bmatrix} n-1 \\ n-2 \end{Bmatrix}.$$

Let us recall

- **RH-X1** $X(\cdot;n) : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-X2** The limits of $X(\zeta;n)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, and are denoted $X_{\pm}(z;n)$ respectively and are related by

$$X_+(z;n) = X_-(z;n)\begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T},$$

- **RH-X3** As $z \to \infty$

$$X(z;n) = \left(I + \frac{\overset{\infty}{X}_1(n)}{z} + \frac{\overset{\infty}{X}_2(n)}{z^2} + O(z^{-3})\right) z^{n\sigma_3}.$$

By $Z(z;n)$ we refer to the solution of the X-RHP when ϕ is replaced by $z\phi$:

- **RH-Z1** $Z(\cdot;n) : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-Z2** The limits of $Z(\zeta;n)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, and are denoted $Z_{\pm}(z;n)$ respectively and are related by

$$Z_+(z;n) = Z_-(z;n)\begin{pmatrix} 1 & z^{-n}z\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T},$$

- **RH-Z3** As $z \to \infty$

$$Z(z;n) = \left(I + O(z^{-1})\right) z^{n\sigma_3}.$$

Theorem

The solution $Z(z;n)$ to the Riemann-Hilbert problem ***RH-Z1*** *through* ***RH-Z3*** *can be expressed in terms of the data extracted from the solution $X(z;n)$ of the Riemann-Hilbert problem* ***RH-X1*** *through* ***RH-X3*** *as*

$$Z(z;n) = \left[\begin{pmatrix} \dfrac{\overset{\infty}{X}_{1,12}(n) X_{21}(0;n)}{X_{11}(0;n)} & -\overset{\infty}{X}_{1,12}(n) \\ -\dfrac{X_{21}(0;n)}{X_{11}(0;n)} & 1 \end{pmatrix} z^{-1} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] X(z;n) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix},$$

or

$$Z(z;n) = \begin{pmatrix} z + \overset{\infty}{X}_{1,22}(n-1) - \dfrac{\overset{\infty}{X}_{2,12}(n-1)}{\overset{\infty}{X}_{1,12}(n-1)} & -\overset{\infty}{X}_{1,12}(n-1) \\ \dfrac{1}{\overset{\infty}{X}_{1,12}(n-1)} & 0 \end{pmatrix} X(z;n-1).$$

$$\mathscr{E}_n[\phi;\psi,\eta;a] := \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+2} & \psi_{n-2} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+3} & \psi_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-3} & \cdots & \phi_0 & \psi_0 \\ \eta_{n-2} & \eta_{n-3} & \cdots & \eta_0 & a \end{pmatrix},$$

$$\mathscr{G}_n[\phi;\psi,\eta;a] := \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+2} & \psi_0 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+3} & \psi_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-3} & \cdots & \phi_0 & \psi_{n-2} \\ \eta_0 & \eta_1 & \cdots & \eta_{n-2} & a \end{pmatrix},$$

$$\mathscr{H}_n[\phi;\psi,\eta;a] := \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+2} & \psi_0 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+3} & \psi_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-3} & \cdots & \phi_0 & \psi_{n-2} \\ \eta_{n-2} & \eta_{n-3} & \cdots & \eta_0 & a \end{pmatrix},$$

$$\mathscr{L}_n[\phi;\psi,\eta;a] := \det\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+2} & \psi_{n-2} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+3} & \psi_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-3} & \cdots & \phi_0 & \psi_0 \\ \eta_0 & \eta_1 & \cdots & \eta_{n-2} & a \end{pmatrix}.$$

$$\underbrace{\mathscr{M}}_{\text{semi-framed}} \cdot \underbrace{\mathscr{M}\begin{Bmatrix} N-2 & N-1 \\ N-2 & N-1 \end{Bmatrix}}_{\text{pure Toeplitz}} = \underbrace{\mathscr{M}\begin{Bmatrix} N-2 \\ N-2 \end{Bmatrix}}_{\text{semi-framed}} \cdot \underbrace{\mathscr{M}\begin{Bmatrix} N-1 \\ N-1 \end{Bmatrix}}_{\text{pure Toeplitz}}$$

$$- \underbrace{\mathscr{M}\begin{Bmatrix} N-2 \\ N-1 \end{Bmatrix}}_{\text{bordered Toeplitz}} \cdot \underbrace{\mathscr{M}\begin{Bmatrix} N-1 \\ N-2 \end{Bmatrix}}_{\text{bordered Toeplitz}} .$$

Theorem

The reproducing kernel $K_n(z_1, z_2) := \sum_{j=0}^{n} Q_j(z_2)\widehat{Q}_j(z_1)$ *has the following semi-framed Toeplitz determinant representation*

$$K_n(z_1, z_2) = a - \frac{1}{D_{n+1}[\phi]} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{1-n} & z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \phi_{n-1} & \cdots & \phi_0 & z_1^n \\ 1 & z_2 & \cdots & z_2^n & a \end{pmatrix}.$$

Theorem

The semi-framed Toeplitz determinants $\mathscr{E}_n[\phi;\psi,\eta;a]$, $\mathscr{G}_n[\phi;\psi,\eta;a]$, $\mathscr{H}_n[\phi;\psi,\eta;a]$ and $\mathscr{L}_n[\phi;\psi,\eta;a]$ can be represented in terms of the reproducing kernel of the system of bi-orthogonal polynomials on the unit circle associated with ϕ given by

$$K_n(z_1,z_2) := \sum_{j=0}^{n} Q_j(z_2)\widehat{Q}_j(z_1),$$

of the system of bi-orthogonal polynomials on the unit circle associated with the symbol ϕ, as

$$\frac{\mathscr{E}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\left[\int_{\mathbb{T}} K_n(z_1,z_2) z_2^{-n}\eta(z_2)\frac{dz_2}{2\pi i z_2}\right] z_1^{-n}\psi(z_1)\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{G}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\left[\int_{\mathbb{T}} K_n(z_1^{-1},z_2^{-1})\eta(z_2)\frac{dz_2}{2\pi i z_2}\right] \psi(z_1)\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{H}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\left[\int_{\mathbb{T}} K_n(z_1^{-1},z_2) z_2^{-n}\eta(z_2)\frac{dz_2}{2\pi i z_2}\right] \psi(z_1)\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{L}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\left[\int_{\mathbb{T}} K_n(z_1,z_2^{-1})\eta(z_2)\frac{dz_2}{2\pi i z_2}\right] z_1^{-n}\psi(z_1)\frac{dz_1}{2\pi i z_1},$$

where $D_n[\phi]$ is given by (2).

Corollary

The semi-framed Toeplitz determinants $\mathscr{H}_{n+2}[\phi;\psi,\eta;a]$, $\mathscr{E}_{n+2}[\phi;\psi,\eta;a]$, $\mathscr{G}_{n+2}[\phi;\psi,\eta;a]$, *and* $\mathscr{L}_{n+2}[\phi;\psi,\eta;a]$ *are encoded into the X-RHP data as*

$$\frac{\mathscr{H}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z_1^{-n} z_2^{-n} \eta(z_2)\psi(z_1)}{z_1 - z_2} \det\begin{pmatrix} X_{11}(z_2;n+1) & X_{21}(z_2;n+2) \\ X_{11}(z_1;n+1) & X_{21}(z_1;n+2)\end{pmatrix} \frac{dz_2}{2\pi i z_2}\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{E}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z_2^{-n} \eta(z_2)\tilde{\psi}(z_1)}{z_1 - z_2} \det\begin{pmatrix} X_{11}(z_2;n+1) & X_{21}(z_2;n+2) \\ X_{11}(z_1;n+1) & X_{21}(z_1;n+2)\end{pmatrix} \frac{dz_2}{2\pi i z_2}\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{G}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z_1^{-n} \tilde{\eta}(z_2)\psi(z_1)}{z_1 - z_2} \det\begin{pmatrix} X_{11}(z_2;n+1) & X_{21}(z_2;n+2) \\ X_{11}(z_1;n+1) & X_{21}(z_1;n+2)\end{pmatrix} \frac{dz_2}{2\pi i z_2}\frac{dz_1}{2\pi i z_1},$$

$$\frac{\mathscr{L}_{n+2}[\phi;\psi,\eta;a]}{D_{n+1}[\phi]} = a - \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\tilde{\eta}(z_2)\tilde{\psi}(z_1)}{z_1 - z_2} \det\begin{pmatrix} X_{11}(z_2;n+1) & X_{21}(z_2;n+2) \\ X_{11}(z_1;n+1) & X_{21}(z_1;n+2)\end{pmatrix} \frac{dz_2}{2\pi i z_2}\frac{dz_1}{2\pi i z_1},$$

where $D_n[\phi]$ *is given by* (2)*, and* X_{11} *and* X_{21} *are respectively the* 11 *and* 21 *entries of the solution to* ***RH-X1*** *through* ***RH-X3****.*

Theorem

Let ϕ be a Szegő-type symbol, and c and d be complex numbers that do not lie on the unit circle. Then, the following Strong Szegő asymptotics hold for $\mathscr{H}$, $\mathscr{L}$, $\mathscr{E}$ and $\mathscr{G}$:

$$\mathscr{H}_{n+1}\left[\phi; \sum_{j=1}^{m_1} \frac{A_j\phi}{z-d_j}, \sum_{k=1}^{m_2} \frac{B_k\phi}{z-c_k}; a\right] = G^n[\phi]E[\phi]\left(a + O(\rho^{-n})\right).$$

$$\mathscr{L}_{n+1}\left[\phi; \sum_{j=1}^{m_1} \frac{A_j\tilde{\phi}}{z-d_j}, \sum_{k=1}^{m_2} \frac{B_k\tilde{\phi}}{z-c_k}; a\right] = G^n[\phi]E[\phi]\left(a + O(\rho^{-n})\right).$$

$$\mathscr{E}_{n+1}\left[\phi; \sum_{j=1}^{m_1} \frac{A_j\tilde{\phi}}{z-d_j}, \sum_{k=1}^{m_2} \frac{B_k\phi}{z-c_k}; a\right] = G^n[\phi]E[\phi]\left(a + \sum_{\substack{j=1 \\ |d_j|<1}}^{m_1} \sum_{\substack{k=1 \\ |c_k|<1}}^{m_2} A_jB_k \frac{\alpha(c_k)}{\alpha(d_j^{-1})} \cdot \frac{1}{1-c_kd_j} + O(\rho^{-n})\right).$$

$$\mathscr{G}_{n+1}\left[\phi; \sum_{j=1}^{m_1} \frac{A_j\phi}{z-d_j}, \sum_{k=1}^{m_2} \frac{B_k\tilde{\phi}}{z-c_k}; a\right] = G^n[\phi]E[\phi]\left(a + \sum_{\substack{j=1 \\ |d_j|<1}}^{m_1} \sum_{\substack{k=1 \\ |c_k|<1}}^{m_2} A_jB_k \frac{\alpha(d_j)}{\alpha(c_k^{-1})} \cdot \frac{1}{1-c_kd_j} + O(\rho^{-n})\right).$$

Theorem

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$$\mathscr{H}_{n+1}\left[\phi;\sum_{j=1}^{m_1}\frac{A_j}{z-d_j},\sum_{k=1}^{m_2}\frac{B_k}{z-c_k};a\right]=G^n[\phi]E[\phi]\left(a+O(\rho^{-n})\right).$$

$$\mathscr{L}_{n+1}\left[\phi;\sum_{j=1}^{m_1}\frac{A_j}{z-d_j},\sum_{k=1}^{m_2}\frac{B_k}{z-c_k};a\right]=G^n[\phi]E[\phi]\left(a+O(\rho^{-n})\right).$$

$$\mathscr{E}_{n+1}\left[\phi;\sum_{j=1}^{m_1}\frac{A_j}{z-d_j},\sum_{k=1}^{m_2}\frac{B_k}{z-c_k};a\right]=G^n[\phi]E[\phi]\left(a+\sum_{\substack{j=1\\|d_j|>1}}^{m_1}\sum_{\substack{k=1\\|c_k|>1}}^{m_2}A_jB_k\frac{\alpha(c_k)}{\alpha(d_j^{-1})}\cdot\frac{1}{1-c_kd_j}+O(\rho^{-n})\right).$$

$$\mathscr{G}_{n+1}\left[\phi;\sum_{j=1}^{m_1}\frac{A_j}{z-d_j},\sum_{k=1}^{m_2}\frac{B_k}{z-c_k};a\right]=G^n[\phi]E[\phi]\left(a+\sum_{\substack{j=1\\|d_j|>1}}^{m_1}\sum_{\substack{k=1\\|c_k|>1}}^{m_2}A_jB_k\frac{\alpha(d_j)}{\alpha(c_k^{-1})}\cdot\frac{1}{1-c_kd_j}+O(\rho^{-n})\right).$$

Consider n simple random walks on $\mathbb{Z}$ which begin at the points $x_1 < x_2 < \cdots < x_n$ and end at the points $y_1 < y_2 < \cdots < y_n$ after a fixed time $T \in 2\mathbb{N}$. If a few of the starting/ending points are not equally spaced, then we get a bordered or framed Toeplitz determinant. For example if $x_j = j$ for $j = 1, \ldots, n$, $y_k = k$ for $j = 1, 2, \ldots, n - m$, and $y_k > n - m$ are arbitrary for $k = n - m + 1, \ldots, n$, we get the bordered Toeplitz determinant

$$\boldsymbol{D}_n^B[\phi; \vec{\boldsymbol{\psi}}_m],$$

where

$$\psi_\ell(\zeta) = \phi(\zeta)\zeta^{-(y_{n-\ell+1}-n)}.$$

If $x_j = y_j = j$ for $j = 1, 2, \cdots, n-1$ and x_n and y_n are both arbitrary then we get the framed Toeplitz matrix

$$\det\begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{n-2} & \psi_{n-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{n-3} & \psi_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{2-n} & \phi_{3-n} & \cdots & \phi_0 & \psi_1 \\ \eta_{1-n} & \eta_{2-n} & \cdots & \eta_{-1} & \phi_{y_n - x_n} \end{pmatrix},$$

where

$$\psi(\zeta) = \phi(\zeta)\zeta^{-(y_n-n)}, \quad \eta(\zeta) = \phi(\zeta)\zeta^{x_n-n}$$

Thank you!

$$\det\begin{pmatrix} T_n[\phi] & B_n \\ C_n & a \end{pmatrix} = D_n[\phi]\,\det\left(a - C_n T_n^{-1}[\phi] B_n\right)$$