A BRIEF INTRODUCTION TO THE CIRCLE METHOD AND SPARSE DOMINATION

and how we use them in the context of discrete harmonic analysis

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Motivation

Improving History

- Area of particular interest since the work of Littman and Strichartz in the early 70s.

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Example

If $p \in (1, 2)$ then the dual index $p' = \dfrac{p}{p-1} \in (2, \infty)$, so an inequality of the form

$$\|Tf\|_{\ell^{p'}(\mathbb{Z})} \leq \|T\| \, \|f\|_{\ell^p(\mathbb{Z})}$$

is an improving inequality for a bounded linear operator T.

Maximal History

- *Hardy and Littlewood Maximal function* for studying cricket averages:

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r) = \{y \in \mathbb{R}^d : |x - y| < r\}$, and $|E|$ denotes the d-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^d$.

The Hardy-Littlewood Maximal Inequality

Theorem (Weak type estimate)

For $d \geq 1$, there is a constant $C_d > 0$ such that for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^d)$, we have:

$$|\{\mathcal{M}f > \lambda\}| \leq \frac{C_d}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}$$

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Theorem (Strong type estimate)

For $d \geq 1$, and $p \in (1, \infty]$, there is a constant $C_{p,d} > 0$ such that for all $f \in L^p(\mathbb{R}^d)$, we have:

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d}\|f\|_{L^p(\mathbb{R}^d)}$$

E. M. Stein showed proved that $C_{p,d} = C_p$.

Other maximal functions in $\mathbb{R}^d$

Let σ be the normalized, rotation invariant measure on $\mathbb{S}^{d-1}$.

- Full Spherical Maximal Function: $\mathcal{M}_{\mathsf{full}} f(x) := \sup_{j \in \mathbb{Z}} \int_{\mathbb{S}^{d-1}} f(x - jy)\, d\sigma(y).$

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- Lacunary Spherical Maximal Function:

  $\mathcal{M}_{\text{lac}} f(x) := \sup_{j \in \mathbb{Z}} \int_{\mathbb{S}^{d-1}} f(x - 2^j y)\, d\sigma(y)$.

  $\mathcal{M}_{\text{lac}}$ is satisfies p bounds for a certain range of p that depends on dimension (different than the one above!). (Lacey, 2017)

Other maximal functions

Maximal operators can be defined for discrete functions f defined on the integers. (two-sided sequences)

Fefferman, C., and E. M. Stein. *"Some Maximal Inequalities."* American Journal of Mathematics, vol. 93, no. 1, 1971, pp. 107–15. JSTOR, https://doi.org/10.2307/2373450.

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A very interesting question is to prove maximal inequalities for various families of averaging operators, both in the continuous and the discrete setting!

Q: How can we achieve that?

Sparse families

Let $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ be a dyadic lattice collection that partitions $\mathbb{R}^d$ into cubes of side-length 2^k and each $Q \in \mathcal{D}_k$ belongs to exactly one cube $R \in \mathcal{D}_{k+1}$.

Definition

A family of cubes $\mathcal{S}$ in $\mathbb{R}^d$ is called 1/2-sparse if for any $Q \in \mathcal{S}$ we have

$$\left| \bigcup_{\substack{Q'\in\mathcal{S},\\ Q'\subsetneq Q}} Q' \right| \leq \frac{1}{2}|Q|.$$

Such a Q' is called a **child** of Q.

Visualization

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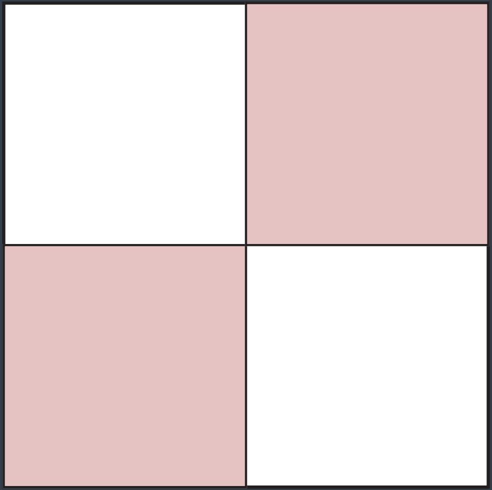


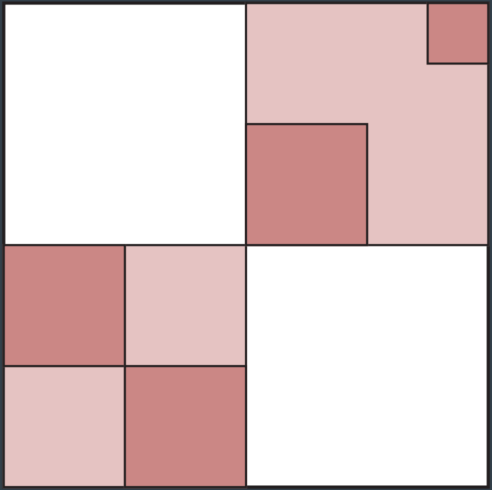
Parent

Q_0

Children

1st generation

Visualization



Parent

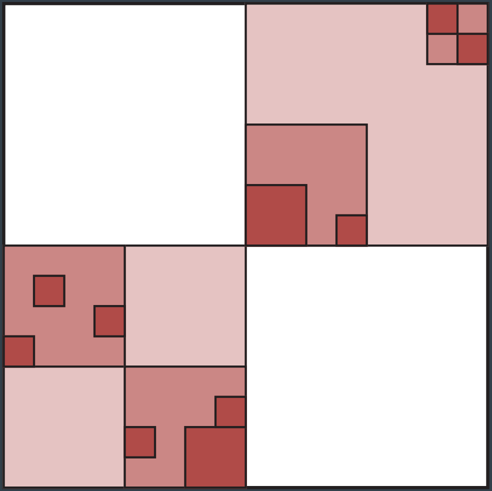
Q_0

Children

1st generation

2nd generation

Visualization



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2nd generation

3rd generation

Exceptional sets

For every cube $Q \in \mathcal{S}$, the exceptional set of Q is

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3. They make up a large portion of Q: $\frac{1}{2}|Q| \leq |E_Q| \leq |Q|$.

Sparse Forms

Given a sparse collection $\mathcal{S}$, and for $1 \leq r, s < \infty$, an (r,s)-sparse form is

$$\Lambda_{r,s}(f,g) = \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{Q,r} \, \langle g \rangle_{Q,s}$$

where

$$\langle f \rangle_{Q,r} = \left(\frac{1}{|Q|} \int_Q |f|^r \right)^{1/r}.$$

When $r = 1$ it is often ommitted.

Why sparse bounds?

Suppose we have an operator T that is dominated by a sparse form:

$$|\langle Tf, g\rangle| \lesssim \sum_{Q\in\mathcal{S}} |Q| \langle f\rangle_Q \langle g\rangle_Q$$

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More generally,

Theorem

If an operator T is dominated by an (r, s) sparse form, then it is bounded $T : L^p(w) \to L^p(w)$, for all weights $w \in A^p$ and $p \in (r, s')$.

where A^p is the Muckenhaupt weight class.

Muckenhaupt Weights

- **Intuitively**, a (positive) weight w is said to be in the p-class of Muckenhaupt weights if the Hardy-Littlewood Maximal Function is bounded in the respective weighted L^p space.

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- **Practically**, the p class of Muckenhaupt weights contains all weights for which w and w^{-1} oscillate in a quantifiably controlled manner. Specifically:

$$\sup_{B\text{: balls}} \left(\frac{1}{|B|} \int_B \left| w(x) \right| dx \right) \left(\frac{1}{|B|} \int_B \frac{1}{\left| w(x) \right|^{p-1}} dx \right)^{\frac{1}{p-1}} < \infty$$

Techniques

The circle method

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Main Idea

Heuristically, when looking at the 1-dimensional torus, we can split it into two pieces called **major** and **minor arcs**, and study their contributions separately. Major arcs contain ‘almost’ rationals whose denominators we can control. Everything else gets thrown in the minor arcs.

We can generalize this idea in higher dimensions!

Formal Definition

Let $0 < \varepsilon \ll 1/4$. For each $s \geq 1$ set

$$\mathcal{R}_s := \left\{ \frac{A}{Q} \in \mathbb{T} : (A, Q) = 1,\ 2^{s-1} \leq Q < 2^s \right\}$$

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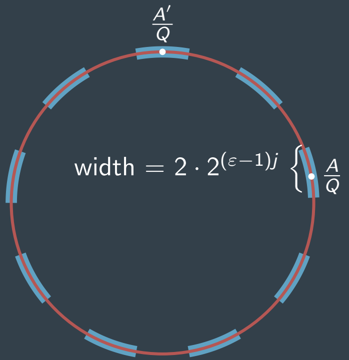
For $s < j\varepsilon$ define the j-th major box at A/Q as

$$\mathfrak{M}_j(A/Q) := \left\{ a \in \mathbb{T} : |a - A/Q| \leq 2^{j(\varepsilon - 1)} \right\}$$

The boxes are disjoint for ε small enough.

Major and Minor arcs



We define the j-th *major arc* as

$$\mathfrak{M}_j := \bigcup_{\frac{A}{Q} \in \mathcal{R}_s} \mathfrak{M}_j(A/Q)$$

Minor arcs are the complements of major arcs.

Fourier Truncation

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Bourgain's High Low Fourier Truncation Method

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Idea: Suppose we want to estimate an inner product. One way is to use Hölder:

$$|\langle f, g\rangle| \leq \|f\|_p \, \|g\|_{p'}.$$

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We break $f = H + L$. That allows us to apply Hölder twice, using different dual indices in each part:

$$|\langle f, g\rangle| = |\langle H + L, g\rangle| \leq \|H\|_p \|g\|_{p'} + \|L\|_q \|g\|_{q'}.$$

Example from Research

Let $d(n)$ be the usual divisor function, and define their sum $D(N) = \sum_{n \leq N} d(n)$.

Now consider the normalized averaging operator

$$K_N f = \frac{1}{D(N)} \sum_{n \leq N} d(n)\, f(x+n)$$

and the associated maximal function

$$K^* f = \sup_N |K_N f|.$$

Results

Theorem (C.G.)

For $p \in (1,2)$, there exists $C_p > 0$ such that for all positive integers N and functions f supported on an interval E of length N, $\langle K_N f\rangle_{E,p'} \leq C_p \langle f\rangle_{E,p}$.

Theorem (C.G.)

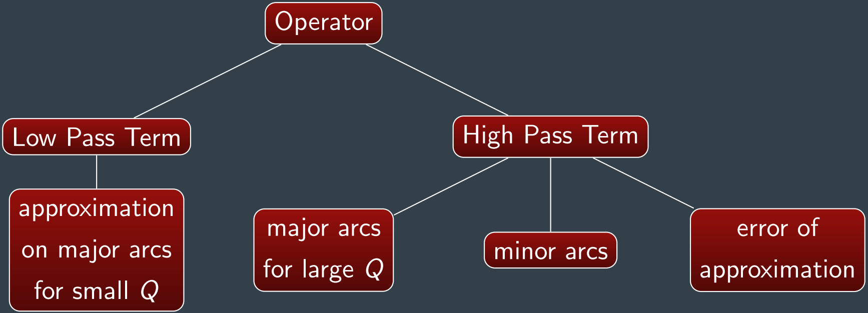
For $r, s \in (1,2)$, there exists $C > 0$ such that for all compactly supported functions f and g, there exists a sparse collection $\mathcal{S}$ so that

$$|(K^* f, g)| \leq C \sum_{I \in \mathcal{S}} |I| \langle f\rangle_{2I,r} \langle g\rangle_{I,s}.$$

Corollary

For any $p \in (1,\infty)$ and all weights w in the Muckenhoupt class A_p, the maximal operator $K^* : \ell^p(w) \to \ell^p(w)$ is a bounded operator.

Decomposition



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Application

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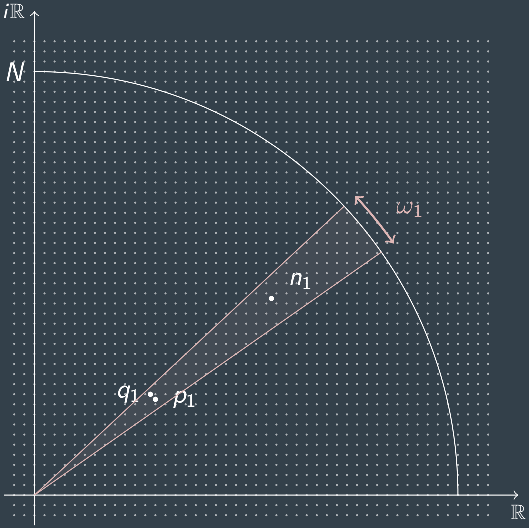

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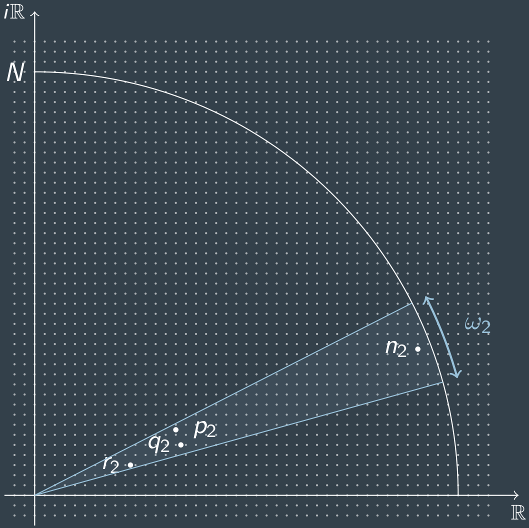

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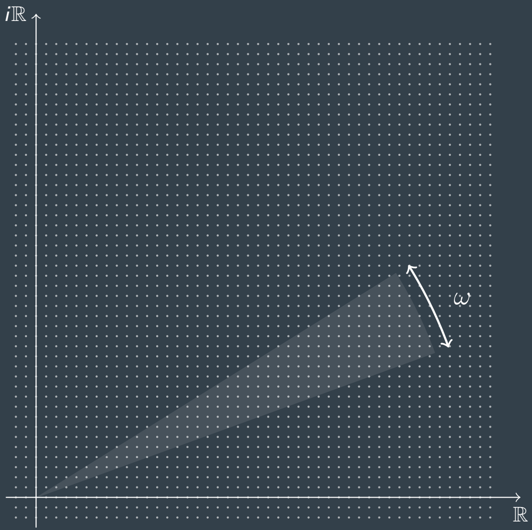
- $\omega_2 = \left[\frac{17}{360}, \frac{24}{360}\right]$

  $n_2 = 76 + 29i = p_2 + q_2 + r_2$

  $$\begin{cases} p_2 = 28 + 13i \\ q_2 = 29 + 10i \\ r_2 = 19 + 6i \end{cases}$$

Generally



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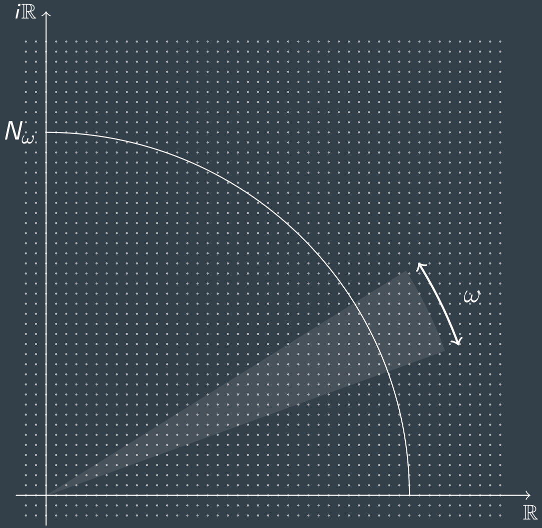

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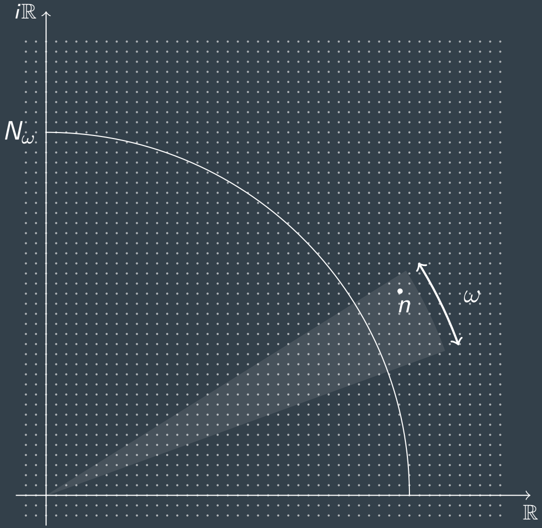

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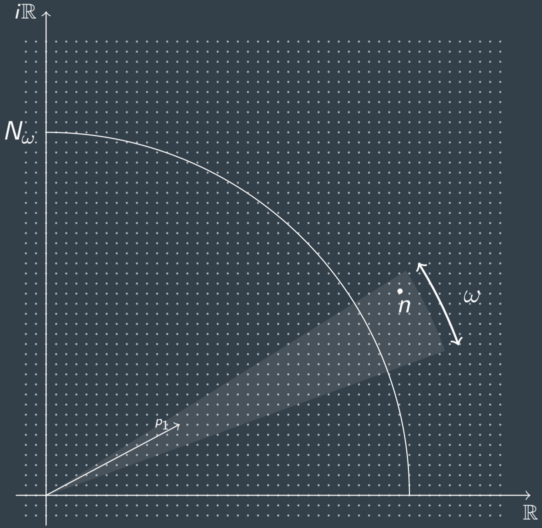

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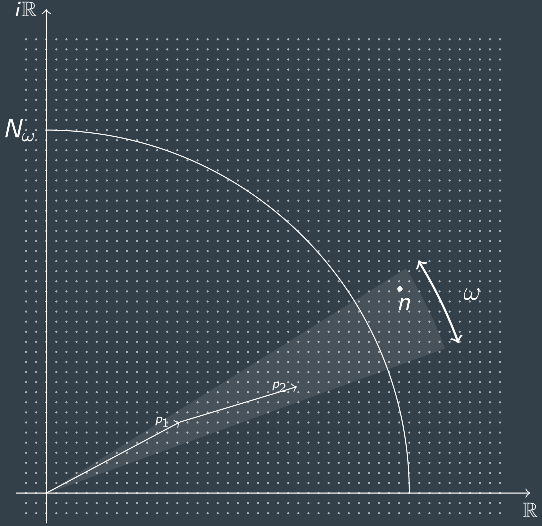

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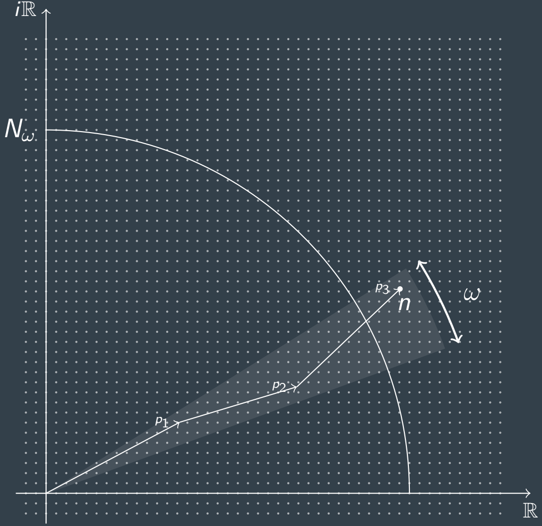

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The end!

Thank you!

Questions?

Let $F, G \subseteq [0,1]$ and $g(x) = x^{-1/4}\mathbb{1}_{[0,1]}$. Then $\langle \mathbb{1}_F \star g, \mathbb{1}_G \rangle \lesssim \big(|F|\,|G|\big)^{7/8}$.