Two-Weight Inequalities for Commutators with Calderón-Zygmund Operators

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Joint work with B. D. Wick, M. Lacey,
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Midwestern Workshop on Asymptotic Analysis
IUPUI, October 6–8, 2017.

Outline

Introduction

Starting point: Coifman, Rochberg and Weiss, *Factorization theorems for Hardy spaces in several variables*, 1976

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- Characterize the norm of the commutator

$$[b, T] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

  where T is a CZO, in terms of the BMO norm of b.

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  where T is a CZO, in terms of the BMO norm of b.

Recall:

- Hilbert transform - $\mathbb{R}$:

$$Hf(x) := \frac{1}{\pi} \text{ p. v. } \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

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Recall:

- Riesz transforms - $\mathbb{R}^n$:

$$R_j f(x) := \frac{\Gamma\left((n+1)/2\right)}{\pi^{(n+1)/2}} \text{ p. v. } \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}}\, dy,$$

$j = 1, \ldots, n.$

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Recall:

- Calderón-Zygmund Operators - $\mathbb{R}^n$:

$$Tf(x) := \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$$

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$$[b, T]f := b(Tf) - T(bf)$$

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- Bounded Mean Oscillation:

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- $\langle b \rangle_Q := \frac{1}{|Q|} \int_Q b(x) \, dx.$
- $H^1(\mathbb{R}^n)$ – $BMO(\mathbb{R}^n)$ Duality (Fefferman, 1971)

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- Two-weight Inequalities: $T : L^p(\mu) \to L^p(\lambda)$

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- Two-weight Inequalities: $T : L^p(\mu) \to L^p(\lambda)$ – much harder!

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- $H : L^p(w) \to L^p(w) \Leftrightarrow w \in A_p$
- A_2 weights:

$$[w]_{A_2} := \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$$

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- What if $\mu \neq \lambda$?

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- What if $\mu \neq \lambda$? Bloom!

Outline

Bloom (1985)

$[b, H]\colon L^p \to L^p$

bounded

$\Longleftrightarrow$

$b \in BMO$

$$\|b\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q \left| b(x) - \langle b \rangle_Q \right| dx$$

Bloom (1985)

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- Extend to all CZO's T on $\mathbb{R}^n$

Bloom (1985)

$[b, H]\colon L^p(\mu) \to L^p(\lambda)$ bounded $\Longleftrightarrow$ $b \in BMO(\nu)$

- Extend to all CZO's T on $\mathbb{R}^n$
- Long-term: Extend to *multiparameter setting*

Bloom (1985)

$[b, H]: L^p(\mu) \to L^p(\lambda)$

bounded

$\Longleftrightarrow$

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- Dyadic approach

Outline

CRW:

Upper Bound:

$$\|[b, T] : L^p \to L^p\| \lesssim \|b\|_{BMO}$$

Lower Bound:

$$\|b\|_{BMO} \lesssim \sum_{j=1}^{n} \|[b, R_j] : L^p \to L^p\|.$$

Main Results (H., Lacey, Wick):

Upper Bound:

$$\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$$

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$$\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$$

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I. Use a Representation Theorem to reduce the problem to bounding

$$[b, \text{Dyadic Shift}]$$

Upper Bound: Strategy

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II. Bound:

$$\|[b, \text{Dyadic Shift}] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$$

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Dyadic Grids:

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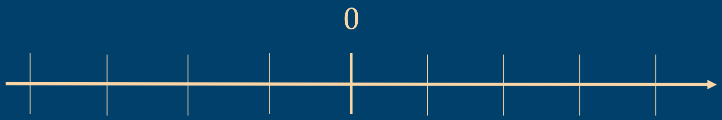
Dyadic Grids: $\mathcal{D}_0$



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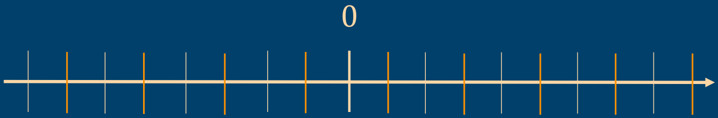

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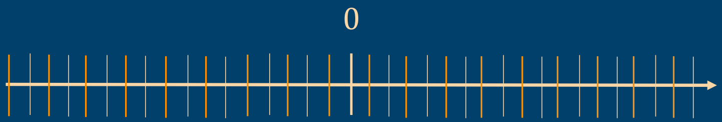

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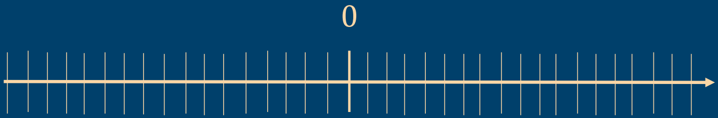

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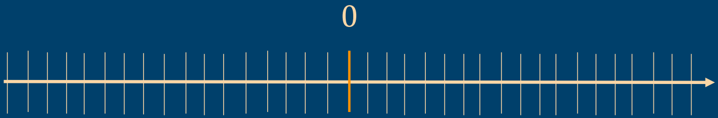
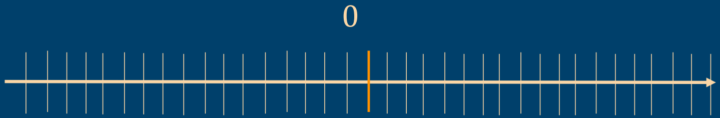


Upper Bound: Strategy

I. Use a Representation Theorem to reduce the problem to bounding

$$[b, \text{Dyadic Shift}]$$

Dyadic Grids: $\mathcal{D}_\omega$



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- $\{I \in \mathcal{D} : |I| = 2^{-k}\}$ forms a partition of $\mathbb{R}$.

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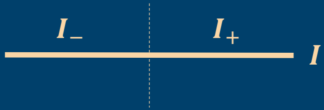
Haar Functions: $I \in \mathcal{D}$

$$h_I := \frac{1}{\sqrt{|I|}} \left(\mathbb{1}_{I_-} - \mathbb{1}_{I_+} \right)$$



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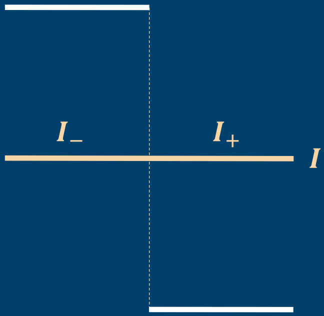

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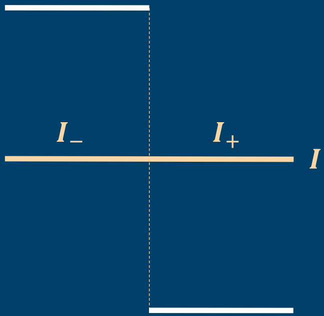

Upper Bound: Strategy

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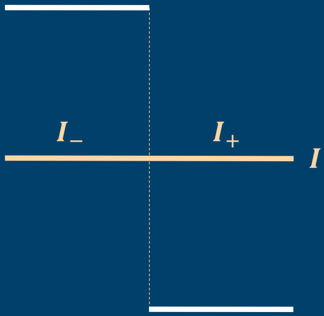
$$f = \sum_{I \in \mathcal{D}} \widehat{f}(I) h_I$$



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Petermichl's Dyadic Shift:

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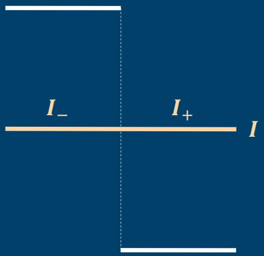

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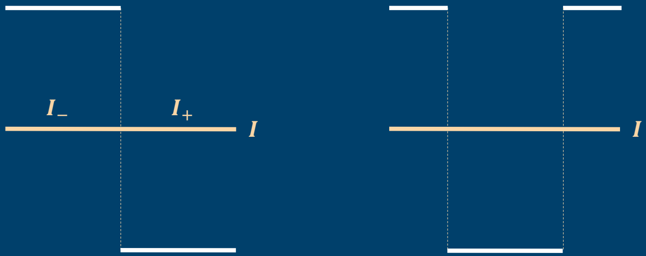

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$$\|[b, Ш_\omega] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$$

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For general CZOs on $\mathbb{R}^n$:

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For general CZOs on $\mathbb{R}^n$: Hytönen Representation Theorem (2011).

Upper Bound: Strategy

II. Bound: $\|[b, \text{Dyadic Shift}] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$

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Paraproducts:

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$$\begin{aligned} [b, Ш]f &= b(Ш f) - Ш(bf) \\ &= (\Pi_b Ш + \Pi_b^* Ш - Ш\Pi_b - Ш\Pi_b^*)f \end{aligned}$$

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Known: $Ш : L^p(w) \to L^p(w)$

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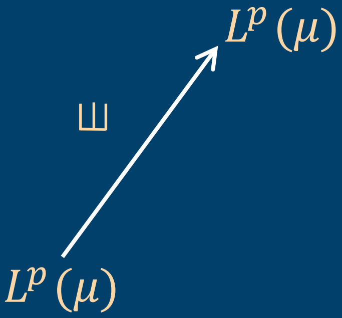

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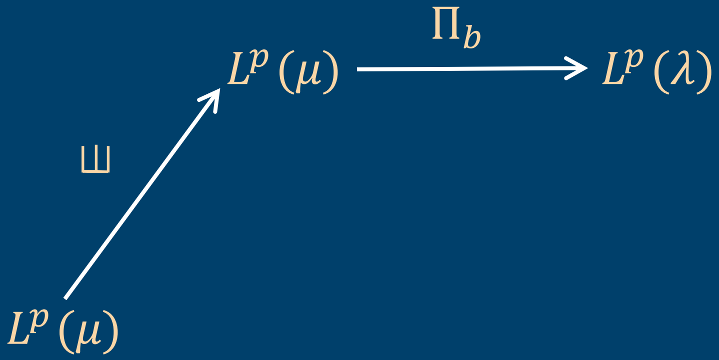

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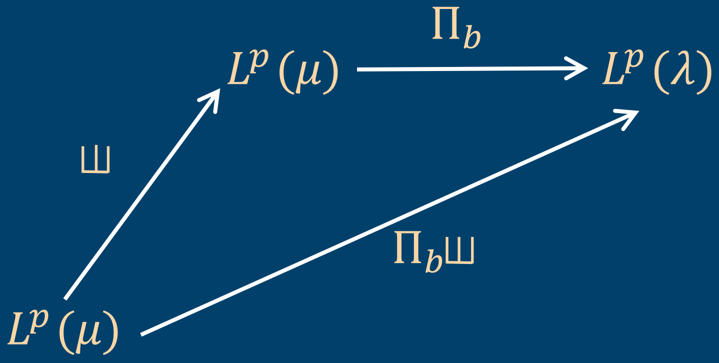

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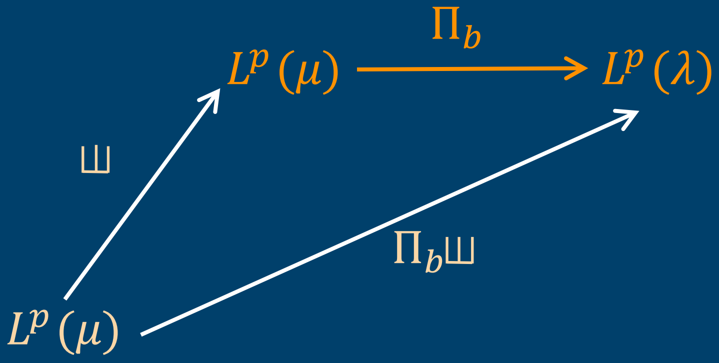

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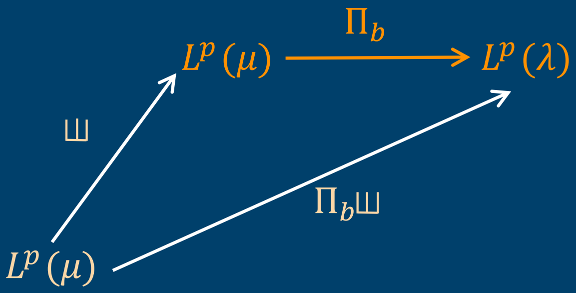

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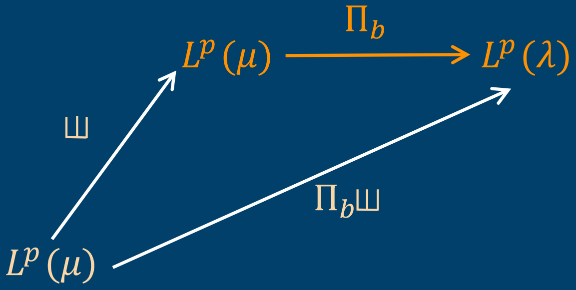
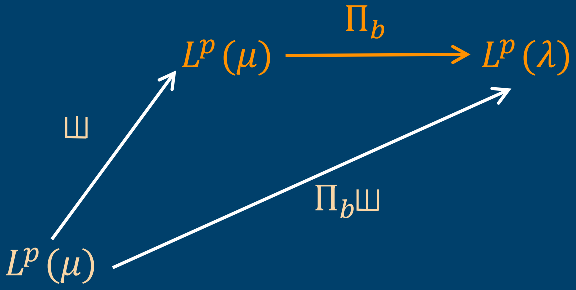


- Reduce to one-weight maximal and square function estimates!

Upper Bound: Strategy



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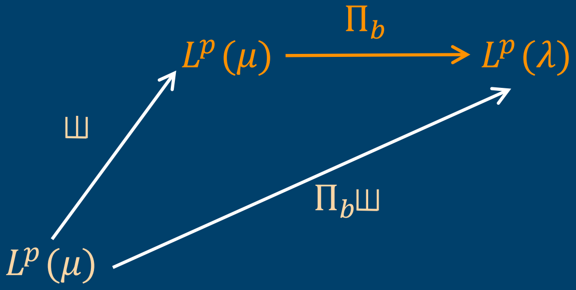

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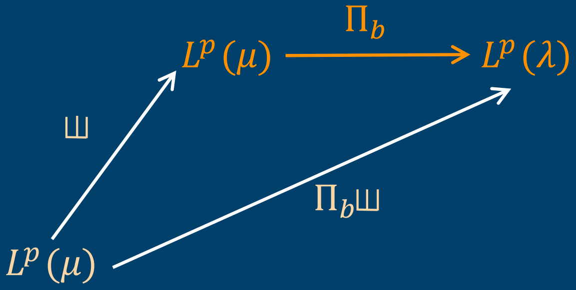

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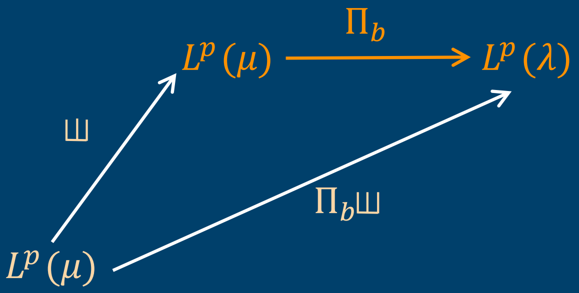

- Reduce to one-weight maximal and square function estimates!
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- $\nu = \mu^{1/p}\lambda^{-1/p} \in A_2$!!!

Outline

Lower Bound

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$$\cong$$

$$\|b\|_{BMO^2(\nu)} := \sup_Q \left(\frac{1}{\nu(Q)} \int_Q |b(x) - \langle b \rangle_Q|^2 \, d\nu^{-1} \right)^{1/2}$$

$$\Downarrow$$

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$\cong$ Muckenhoupt & Wheeden ('75)

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⬇

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