On a problem of E. Meckes for the unitary eigenvalue process on an arc

Midwestern Workshop on Asymptotic Analysis

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Unitary eigenvalue process

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Fix $\theta \in (0, 2\pi)$, and let

$$\mathcal{N}_\theta := \#\{j : 0 < \theta_j < \theta\}.$$

Unitary eigenvalue process

The set of eigenvalues is a *determinantal point process*, meaning that there exists a kernel $K_n : [0, 2\pi] \times [0, 2\pi] \to [0, 1]$ such that, for pairwise disjoint subsets $A_1, ..., A_k \subset [0, 2\pi]$,

$$\mathbb{E}\left[\prod_{j=1}^{k} \mathcal{N}_{A_i}\right] = \int_{A_1} ... \int_{A_k} \det[K_n(x_i, x_j)]_{i,j=1}^{k} d\mu(x_1)...d\mu(x_k),$$

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where the kernel K_n is given by

$$K_n(x, y) := \begin{cases} \sin(\frac{n(x-y)}{2}) / \sin(\frac{(x-y)}{2}), & \text{if } x - y \neq 0, \\ n, & \text{if } x - y = 0. \end{cases}$$

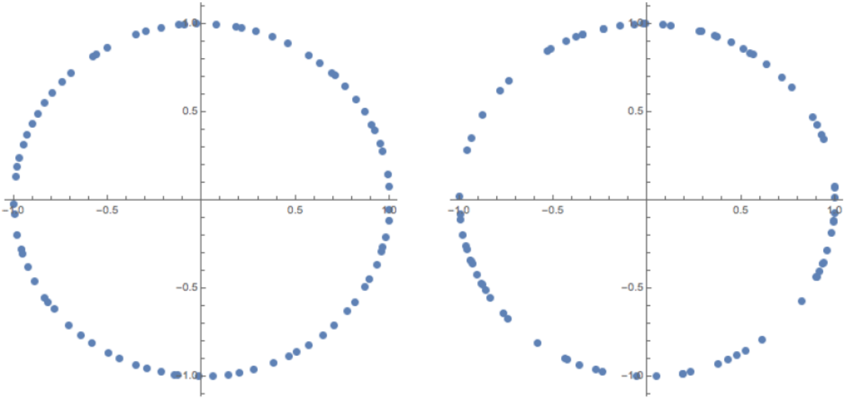

Figure 3.1 On the left are the eigenvalues of an 80×80 random unitary matrix; on the right are 80 i.i.d. uniform random points.

Theorem (J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, [1])

Let $\mathcal{X}$ be a DPP on a compact metric measure space (Λ, μ) with kernel $K : \Lambda \times \Lambda \to \mathbb{C}$. Suppose that

$$\mathcal{K}(f)(x) = \int K(x,y)f(y)d\mu(y), \quad f \in L^2(\mu)$$

is self-adjoint, nonnegative, and locally trace-class with eigenvalues in [0,1].

Let $K_D(x,y) = \mathbb{I}_D(x)K(x,y)\mathbb{I}_D(y)$ be the restriction of K to $D \subset \Lambda$.
Denote by $\{p_j\}_{j\in\mathcal{A}}$ the eigenvalues of $\mathcal{K}_D(x,y)$ and by $\mathcal{N}_D$ the number of particles of the DPP which lie in D.
Then

$$\mathcal{N}_D \stackrel{d}{=} \sum_{j\in\mathcal{A}} \xi_j,$$

where ξ_j are independent Bernoulli random variables with $P[\xi_j = 1] = p_j$ and $P[\xi_j = 0] = 1 - p_j$.

Motivation: Meckes' problem

By the theorem above,

$$\mathcal{N}_\theta \stackrel{d}{=} \sum_{j=1}^{n} \xi_j,$$

where $\mathbb{P}[\xi_j = 1] = p_j$ and $\mathbb{P}[\xi_j = 0] = 1 - p_j$.

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Question (E. Meckes)

What are the asymptotics for p_j near zero?

Lemma (D. Slepian, 1978)

i) for any fixed $\rho \in (0,1)$, *there exist constants* $c_0 = c_0(\rho)$, $n_0 = n_0(\rho)$ *such that*

$$p_j(n) \geq 1 - e^{-c_0 n} \text{ for all } j \leq \frac{n\theta}{2\pi}(1-\rho) \text{ and all } n \geq n_0;$$

ii) for any fixed $\rho \in (0, 2\pi/\theta - 1)$ *there exist constants* $c_1 = c_1(\rho)$, $n_1 = n_1(\rho)$ *such that*

$$p_j(n) \leq e^{-c_1 n} \text{ for all } j \geq \frac{n\theta}{2\pi}(1+\rho) \text{ and all } n \geq n_1.$$

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$$p_j(n) \leq e^{-c_1 n} \text{ for all } j \geq \frac{n\theta}{2\pi}(1+\rho) \text{ and all } n \geq n_1.$$

For K large and fixed, and $\lambda \in \mathbb{R}$, consider

$$G(\lambda, n) := \#\{j : p_j > Ke^{-\lambda n}\}.$$

Our goal is to understand $G(\lambda, n)$ as a function of λ and n.

Theorem (K.-Saff, 2023)

For any fixed $\varepsilon > 0$*,*

$$\frac{1}{2\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |G(x,n)|dx = \frac{n}{2\varepsilon}(\Lambda(\lambda+\varepsilon) - \Lambda(\lambda-\varepsilon)) - o(n),$$

where the function $\Lambda(\lambda) = \sup_{q\in[0,1]} \{q\lambda - I(q)\}$ *is given by the Fenchel-Legendre transform of the function* I*.*
In particular, if $\lambda \geq C$*, where* C *is explicitly known constant, then*

$$\frac{1}{2\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |G(x,n)|dx = n - o(n).$$

Large deviation principle

Definition

A sequence of Borel measures $\{P_n\}$ on a topological space X satisfies a large deviation principle (LDP) with rate function I and speed s_n if for all Borel sets $\mathcal{B} \subseteq X$,

$$-\inf_{x\in\mathcal{B}^0} I(x) \le \liminf_{n\to\infty} \frac{1}{s_n} \log(P_n(\mathcal{B})) \le \limsup_{n\to\infty} \frac{1}{s_n} \log(P_n(\mathcal{B})) \le -\inf_{x\in\overline{\mathcal{B}}} I(x)$$

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Example. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. with law P and mean m. Then, by the law of large numbers,

$$S_n := \frac{X_1 + X_2 + \ldots + X_n}{n} \underset{n\to\infty}{\longrightarrow} m.$$

For $x \geq m$ we get

$$\mathbb{P}(S_n \geq x) \leq e^{-n\Phi(x)},$$

where $\Phi(x) = \sup_{\beta\in\mathbb{R}}(\beta x - \log\phi(\beta))$, and $\phi(\beta) = \int e^{\beta x} dP$.

Theorem (F. Hiai, D. Petz)

Let $U_n \in \mathbb{U}(n)$ and $\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{e^{i\theta_j}}$, where $\{e^{i\theta_j}\}_{j=1}^{n}$ are the eigenvalues of U_n. Denote by P_n the law of μ_n. Then the sequence $\{P_n\}$ satisfies an LDP on the space $\mathcal{P}(\mathbb{S}^1)$ of probability measures on the unit circle equipped with the topology of weak convergence, with speed n^2 and strictly convex rate function

$$\mathcal{E}(\nu) = -\iint_{\mathbb{S}^1\times\mathbb{S}^1} \log|z-w| d\nu(z) d\nu(w).$$

Connection to the constrained energy problem

From Hiai-Petz theorem it follows that the random variables $\mu_n(A_\theta) = \frac{N_\theta}{n}$ satisfies an *LDP* on $[0,1]$ with speed n^2 and rate function

$$I(q) := \inf\{\mathcal{E}(\nu) : \nu \in \mathcal{P}(\mathbb{S}^1), \nu(A_\theta) = q\},$$

where A_θ is an arc from $e^{-i\theta/2}$ to $e^{i\theta/2}$.

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On the other hand, we have

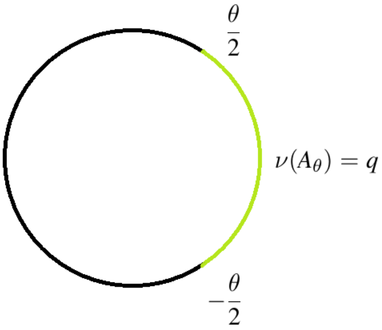
$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{E}[e^{\lambda n N_\theta}] = \lim_{n\to\infty} \int_0^\lambda \frac{G(x,n)}{n} dx = \sup_{q\in[0,1]} (q\lambda - I(q)),$$

where

$$G(\lambda, n) := \#\{j : p_j > Ke^{-\lambda n}\}.$$

Problem I

Given q and θ, with $0 < q < 1$, $0 < \theta < 2\pi$, determine a measure $\nu \in \mathcal{P}(\mathbb{S}^1)$ that minimizes the energy $\mathcal{E}(\nu)$, subject to constraint $\nu(A_\theta) = q$.



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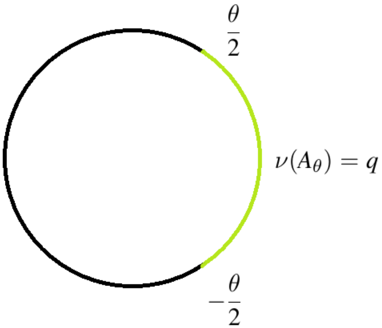
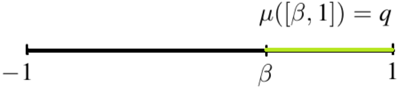
Problem II

Given q and β, with $0 < q < 1$, $-1 < \beta < 1$, determine a measure $\mu \in \mathcal{P}([-1, 1])$ that minimizes the energy $\mathcal{E}(\mu)$, subject to constraint $\mu([\beta, 1]) = q$.





The limiting cases

- When $\theta \to 0$, the Problem I becomes the weighted energy problem on the unit circle with an external field $Q(z) = \frac{q}{1-q} \log \frac{1}{|z-1|}$. (Lachance, Saff, Varga, '79)
- When $\beta \to 1$, the Problem II becomes the weighted energy problem on $[-1, 1]$ with an external field $Q(z) = \frac{q}{1-q} \log \frac{1}{|z-1|}$. (Saff, Ullman, Varga, '80)

The limiting cases

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In particular, when a charge amount $q > 0$ is placed at $t = 1$, the equilibrium charge distribution of amount $1 - q$ on $[-1, 1]$ is given by

$$d\mu^*(x) = \frac{\sqrt{|x-\alpha|}}{\pi\sqrt{(x+1)(1-x)}}dx, \quad x \in [-1, \alpha],$$

where $\alpha = 1 - 2q^2$.

Theorem 1. (K.-Saff, 2023)
The measure $\nu^* \in \mathcal{P}(\mathbb{S}^1)$ such that
$\mathcal{E}(\nu^*) = \inf\{\mathcal{E}(\nu) : \nu \in \mathcal{P}(\mathbb{S}^1), \nu(A_\theta) = q\}$, is unique and
i) if $q \geq \frac{\theta}{2\pi}$, is given by

$$d\nu^*(e^{i\psi}) = \frac{\sqrt{|\cos(\psi) - \alpha|}}{2\pi\sqrt{|\cos(\psi) - \cos(\frac{\theta}{2})|}} d\psi, \tag{3}$$

where $e^{i\psi} \in A_\theta \cup \{z \in \mathbb{S}^1 : \arccos(\alpha) \leq \arg\ z \leq 2\pi - \arccos(\alpha)\}$ and with α determined from the equation

$$\int_{-1}^{\alpha} \frac{\sqrt{|x-\alpha|}}{\pi\sqrt{|(x+1)(x-\cos(\frac{\theta}{2}))(x-1)|}} dx = 1 - q;$$

ii) if $q \leq \frac{\theta}{2\pi}$, is given by (2), where
$e^{i\psi} \in A_\theta^c \cup \{z \in \mathbb{S}^1 : -\arccos(\alpha) \leq \arg\ z \leq \arccos(\alpha)\}$ and α is a solution to the equation

$$\int_{-1}^{\beta} \frac{\sqrt{|x-\alpha|}}{\pi\sqrt{|(x+1)(x-\cos(\frac{\theta}{2}))(x-1)|}} dx = 1 - q.$$

Theorem 2. (K.-E.B.Saff, 2023), (A. Martínez-Finkelshtein, E.B.Saff, 2002)
The measure $\mu^* \in \mathcal{P}([-1,1])$ such that
$\mathcal{E}(\mu^*) = \inf\{\mathcal{E}(\mu) : \mu \in \mathcal{P}([-1,1]), \mu([\beta,1]) = q\}$, is unique and
i) if $q \geq \frac{1}{\pi}\int_\beta^1 \frac{1}{\sqrt{1-x^2}}dx$, is given by

$$d\mu^*(x) = \frac{\sqrt{|x-\alpha|}}{\pi\sqrt{|(x+1)(x-\beta)(x-1)|}}dx, \tag{2}$$

where $x \in [-1,\alpha] \cup [\beta,1]$ and α is determined from the equation

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Energy problem with prescribed masses

Suppose Σ_1, Σ_2 are closed disjoint sets on $\mathbb{C}$ of **positive distance** from one another. We want to minimize the energy

$$\iint \log \frac{1}{|z-\zeta|} d\sigma(z) d\sigma(\zeta) \tag{4}$$

for all measures σ of the from $\sigma = \sigma_1 + \sigma_2$, where σ_j is a compactly supported measure of total mass m_j on Σ_j.

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For $z \in \Sigma_j$, set

$$w_j^{\sigma}(z) := \exp(-U^{\overline{\sigma}_j}(z)/m_j), \quad j = \overline{1,2}$$

where

$$U^{\overline{\sigma}_j}(z) := \int \log \frac{1}{|z-\zeta|} d\overline{\sigma}_j(\zeta), \qquad \overline{\sigma}_1 := \sigma_2, \qquad \overline{\sigma}_2 := \sigma_1.$$

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We call by $\mu^* = \mu_1^* + \mu_2^*$ the measure minimizing (4).

Theorem (Characterization of the optimal measure on $\Sigma_1 \cup \Sigma_2$*)

For $j = 1, 2$ we have

$$\mu_j^* = m_j \mu_{w_j^{(\mu^*)}},$$

where $\mu_{w_j^{(\mu^*)}}$ is the unit measure that is optimal for the weighted energy problem on Σ_j corresponding to $w_j^{(\mu^*)}$.

Conversely, if for some σ supported on $\Sigma_1 \cup \Sigma_2$ with $\|\sigma\|_{\Sigma_1} = m_1$, $\|\sigma\|_{\Sigma_2} = m_2$ we have

$$\sigma_j = m_j \mu_{w_j^{(\sigma)}}, \quad j = 1, 2,$$

then $\sigma = \mu^*$.

*Special case of Theorem VIII.2.1 from the book by Saff-Totik.

Frostman inequalities

Thus, if μ^* is an optimal measure, there exist constants F_1, F_2 such that

$$U^{\mu^*}(z) \geq F_1, \ \text{ q.e. on } \Sigma_1, \qquad U^{\mu^*}(z) = F_1, \ \text{ q.e. on supp } \mu_1^*,$$

$$U^{\mu^*}(z) \geq F_2, \ \text{ q.e. on } \Sigma_2, \qquad U^{\mu^*}(z) = F_2, \ \text{ q.e. on supp } \mu_2^*.$$

Constrained problem on an interval. Determining the support of μ^*

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Notice that if $q = \frac{1}{\pi}\int_\beta^1 \frac{1}{\sqrt{1-x^2}}dx$, then

$$d\mu(x) = \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}, \quad x \in [-1,1].$$

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$$d\mu(x) = \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}, \quad x \in [-1, 1].$$

In the case $q \neq \frac{1}{\pi}\int_\beta^1 \frac{1}{\sqrt{1-x^2}}dx$ the support of μ^* is $[-1, \alpha_0] \cup [\beta_0, 1]$, $\alpha_0 < \beta_0$. Indeed, supp $\mu_1^* \cap (-1, \beta)$ is an interval due to the fact that μ_1^* is the solution to the equilibrium problem on $[-1, \beta]$ with the convex external field $U^{\mu_2^*}(z)$. Similarly, supp $\mu_2^* \cap (\beta, 1)$ is an interval.

Constrained problem on an interval. Finding the density function of μ^*.

Consider

$$H(z) = \int \frac{d\mu^*(\zeta)}{z - \zeta}.$$

on the Riemann sphere $\overline{\mathbb{C}}$ cut along the support of μ^*, $[-1, \alpha_0] \cup [\beta_0, 1]$.

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$H^2(z)$ is a rational function on $\overline{\mathbb{C}}$ with at most simple poles at the points $\{-1, \alpha_0, \beta_0, 1\}$ and $H^2(z) \sim \frac{1}{z^2}$ when $z \to \infty$. Thus,

$$H^2(z) = \frac{(z-A)(z-B)}{(z+1)(z-\alpha_0)(z-\beta_0)(z-1)}, \quad A, B \in \mathbb{R},$$

$$H(z) = \frac{i|z-A||z-B|}{\sqrt{(z+1)(z-\alpha_0)(z-\beta_0)(z-1)}}, \quad z \in \text{supp } \mu^*.$$

Constrained problem on an interval. Finding the density function of μ^*.

Cauchy's formula gives

$$H(z) = \frac{1}{2\pi i} \oint_{\text{supp } \mu^*} \frac{H(\zeta)}{\zeta - z} d\zeta = \frac{1}{\pi i} \int_{\text{supp } \mu^*} \frac{H(y)}{y - z} dy,$$

and since $H(z) = \int \frac{d\mu^*(\zeta)}{z - \zeta}$, we have

$$d\mu^*(y) = \frac{|y - A||y - B|}{\pi \sqrt{(y+1)(y-\alpha_0)(y-\beta_0)(y-1)}} dy, \quad A, B \in \mathbb{R}$$

Next, we show that $A = \alpha_0$, $\beta_0 = \beta$ if $q > \frac{1}{\pi} \int_\beta^1 \frac{1}{\sqrt{1-x^2}} dx$.

Constrained problem on an interval. Finding the density function of μ^*.

For $x \in (\alpha_0, \beta_0)$ consider

$$\frac{dU^{\mu^*}(x)}{dx} = -\frac{1}{\pi} \int_{[-1,\alpha_0]\cup[\beta_0,1]} \frac{1}{x-y} \frac{|y-A||y-B|}{\sqrt{(y+1)(y-\alpha_0)(y-\beta_0)(y-1)}} dy,$$

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To prove the above claims, recall that we have

$$U^{\mu^*}(z) \geq F_1, \text{ q.e. on } \Sigma_1, \qquad U^{\mu^*}(z) = F_1, \text{ q.e. on supp } \mu_1^*,$$

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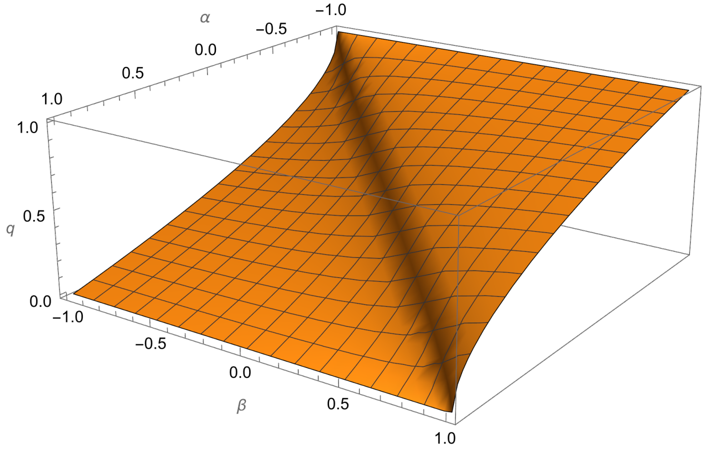

Figure: Graph showing the relationship between the parameters α, β and q.

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We show that

$$U^{\mu^*}(\Psi(e^{i\varphi})) = 2U^{\nu^*}(e^{i\varphi}) + \log 2,$$

where μ^* is the solution to the Problem II with $\beta = \cos(\frac{\theta}{2})$, and conclude from here that ν^* is optimal.

Thank you!

J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Determinantal processes and independence*, Probab. Surv. 3 (2006), 206–229. MR-2216966.

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