Quadrature Domains in Complex Variables

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Usual test class is harmonic functions, but we want to use tools of complex analysis. So we'll use the Bergman Space (square-integrable holomorphic functions.)

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The points are called 'quadrature nodes' and the integration formula is called a 'quadrature identity (QI).'

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- Very nice connections have been found to fluid flow, free boundaries, subnormal operators, potential theory, Riemann surfaces...
- Building on some of this research, Bell discovered that he could synthesize much of the introductory theory by using QI's which are valid for H^2, by using the L^2 tools of complex analysis (i.e. the Bergman kernel and projection).

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- The premium example is the disc. The harmonic mean value theorem says that integration is a multiple of point evaluation at the center. For example, $\int_{\mathbb{D}} f(z)dA = \pi f(0)$.

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- The cardioid (quadratic image of a disc) is an example with one node and two terms in the QI: the QI involves evaluating a function at the node and evaluating the derivative at the node.
- The Neumann oval is another 'order 2' example: there are two nodes, and in the QI the function is evaluated at each node. (Neumann oval is the inversion of the exterior of an ellipse through a circle).

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- Or you can think about 'sweeping' the measure involved onto the quadrature nodes.

- Analytic continuation: Consider $\bar{z}$ defined on $bd(\Omega)$. If Ω is real-analytic, Cauchy-Kovalevskaya says $\bar{z}$ extends analytically to a neighborhood of $bd(\Omega)$. In that case, being a QD for holomorphic functions means that the function $\bar{z}$ extends analytically all the way inside Ω, except at the nodes. In other words it extends meromorphically inside.

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- Then consider the Schottky double. Glue the domain to a copy of itself along the boundary. Now you can proceed with some Riemann surface theory.
- For example, now $z, \bar{z}$ extend from the boundary meromorphically to the double; that means they depend polynomially on one other on the boundary. So the boundary of a QD is algebraic in the plane!

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What about higher dimensions?

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Answer: We don't know too much. Some has been written about QD's for harmonic functions. (E.g. Lundberg and Eremenko showed their boundaries can be worse than in the plane).

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For holomorphic functions, Bell has a few ruminations, and Haridas and Verma have a paper about approximating certain product domains by quadrature domains.

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The motivation is to start by seeing how elegant results in the plane might extend to several variables, by using the Bergman kernel and projection.
So, we'll start with some more overview of why QD's for H^2 functions are beautiful in the plane. Later, I'll show the progress I've made in pushing those ideas into $\mathbb{C}^n$.
And I'll point out when things don't look too pretty.

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- But also, $\int f = \int f \cdot 1$; whereas a QI would mean $\int f = \sum_{i\alpha} c_{i\alpha} f^\alpha(w_i)$
- that means Ω is a QD if and only if the function $1 \in H^2(\Omega)$ is a linear combination of the Bergman kernel and its derivatives at some points w_i. (Here as always, I'm going to assume Ω has finite volume)

Background: Bergman Span

We frame that idea in a definition:

Definition The Bergman span of $\Omega \subset \mathbb{C}^n$, a domain with finite volume, is the linear span of the functions $\frac{\partial^\alpha K(z,w)}{\partial \bar{w}^\alpha}|_{w_0}$ as α varies over multiindices and w_0 varies over Ω.

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Conclusion: Ω of finite volume is a QD if and only if the function 1 is a member of the Bergman span of Ω.

Background: Disc Properties

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- Conformal maps to other simply connected domains: Riemann Mapping Theorem

Background: Disc-like Properties

QD's for H^2 functions imitate the disc in many ways, and have impressive conformal mapping properties. For us, the relevant properties are:

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- Bergman span contains all polynomials
- Conformal maps between QD's are algebraic
- The only bounded simply connected QD's are rational images of the disc
- Any smooth bounded domain is conformally equivalent to a smooth QD which is arbitrarily $\mathcal{C}^\infty$ close to the domain

Background: Mapping Properties

There are two nice criteria for knowing when a conformal map lands you in a QD:

- If Ω has finite area and $f : \Omega \to V$ is a biholomorphism, then V is a QD if and only if f' is in the Bergman span of Ω.
- For nice bounded domains, the image of a conformal map is a QD exactly when it is a ratio of Bergman span elements.

Several Dimensions

Those are the motivational properties in the plane. A couple of them travel to $\mathbb{C}^n$ with no trouble, as noted by Bell:

- If $f : \Omega \to V$ is a biholomorphism of finite-volume domains in $\mathbb{C}^n$, then V is a QD if and only if $J_{\mathbb{C}} f$ is in the Bergman span of Ω.

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- Ω is a QD if and only if 1 is in the Bergman span of Ω.
- Why? The same reasoning as in the plane works for the characterization of 1 being in the Bergman span. From there:
- If $f : \Omega \to V$ is a biholomorphism, then $h \to (J_{\mathbb{C}} f) \cdot h \circ f$, is a Bergman Space isomorphism $H^2(V) \to H^2(\Omega)$. It's also a one-to-one correspondence between the Bergman spans of Ω and V. You can 'push forward' and 'pull back' to switch between the Bergman spans of Ω and V. The function 1 corresponds to $J_{\mathbb{C}} f$ under this correspondence.

Let's start investigating several dimensions.

QDP's

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Fact: the property that all polynomials reside in the Bergman Span on a QD in the plane fails miserably in higher dimensions, as we'll see later. Forcing them to be there after all is a good way to get some mapping properties; we make a definition:

Definition A QDP is a quadrature domain in $\mathbb{C}^n$ whose Bergman span contains all holomorphic polynomials.

First Results

Some results for QDP's:

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- Generalization from the disc: A product of bounded simply connected domains $\Pi_1^n \Omega_j$ is a QDP if and only if it is a rational image of the unit polydisc. (Exploit the automorphism structure of the polydisc together with the Riemann mapping theorem)
- Another mapping result from the plane generalizes: If $f : \Omega \to V$ is biholomorphic, where Ω is a product domain and a QDP, and V any QDP, then f must be algebraic.

Unit Ball generalization

- If Ω is any domain and $f : \Omega \to V$ is a biholomorphism, and V is a QDP, then f is a Bergman Coordinate of Ω (all its component functions are ratios of Bergman span elements).
- That's because 1 and ζ_j are in the Bergman span of V (ζ being the coordinate on V), and they pull back to $J_{\mathbb{C}}f$ and $J_{\mathbb{C}}f \cdot f_j$. So both those are in the Bergman span of Ω; now divide them.

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Corollary another nice generalization: if $f : \mathbb{B} \to V$ and V is a QDP, then f must be rational, where $\mathbb{B}$ is the unit ball.

- This follows since the Bergman kernel of the ball is rational, and any map to a QDP must be a quotient of rational functions.

Circular Domains

Circular domains have a very good Bergman span: if Ω is bounded, circular, and contains 0, then Ω is a QDP. In fact, inner products with any polynomials give function and/or derivative values at 0. This leads to another mapping result:

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- If Ω is circular, bounded, contains 0, and $f : \Omega \to V$ is biholomorphic, then f is a polynomial mapping if and only if V is a QDP with the special property that every polynomial comes from the Bergman span elements corresponding to the point $f(0)$.

What next?

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We've seen that requiring all polynomials to reside in the Bergman span provides just enough extra structure beyond the definition of QD to recover some properties similar to the plane.
However, not every QD is a QDP in several variables. What types of properties can we look for if we don't have polynomials to rely on?

Smooth Density

One property we can try to recover without polynomials is 'QD density'. Recall that any smooth bounded planar domain is approximable by smooth quadrature domains.

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One property we can try to recover without polynomials is 'QD density'. Recall that any smooth bounded planar domain is approximable by smooth quadrature domains.
For the special case of smooth bounded convex domains in $\mathbb{C}^n$, I've made some progress in this direction.

A mapping theorem

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Theorem

$\Omega \subset \mathbb{C}^n$ must be biholomorphic to a QD if it meets the following restrictions:

- *Ω is smooth and bounded*
- *Ω satisfies Condition R (Bergman projection is regular up to the boundary)*
- *Ω's projection onto the first $n-1$ dimensions is pseudoconvex.*
- *Ω's cross sections in the n^{th} coordinate are all convex*
- *Ω contains the graph of a smooth function over the first $n-1$ coordinates.*

A few words about the proof

Why the weird conditions?

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- So the search for a QD biholomorphically equivalent to Ω is the same as the search for a one-to-one map whose Jacobian is in the Bergman span.

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- So the search for a QD biholomorphically equivalent to Ω is the same as the search for a one-to-one map whose Jacobian is in the Bergman span.
- Density of Bergman span comes from Condition R, so choose a Bergman span element g smoothly close to 1, then try to antidifferentiate it in the z_n direction. Call the z_n-antiderivative G.
- Make up a mapping $f = (z_1, z_2, \cdots, z_{n-1}, G)$. Show it's one-to-one; note that its Jacobian determinant is $1 \cdot 1 \cdots 1 \cdots 1 \cdot \frac{\partial G}{\partial \bar{z}_n} = g$, which is in the Bergman span.

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- The criteria needed to pull off that stunt are the conditions in the theorem.

Smooth approximation by QD's

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It turns out that smooth bounded convex domains meet all the requirements. So they're all biholomorphic to quadrature domains. And it gets better! If the smooth function whose graph is contained in your domain can be chosen holomorphic, then the proof ends up showing that the domain is smoothly approximable by QD's. For example, any smooth bounded convex domain symmetric about $\{z_n = 0\}$ is smoothly approximable by QD's.

Counterpoint

Next, we'll take a look at some properties that don't transfer well to several dimensions.

- First is the fact that's been alluded to already, that not every QD will contain every polynomial in its Bergman span. In fact, a QD can include infinitely many polynomials in its Bergman span, and exclude infinitely many others.

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You can map the polydisc with a bad-looking mapping whose Jacobian nevertheless is just 1. The image is a QD since the Jacobian is in the Bergman span, but the badness of the mapping itself will prevent some polynomials from ending up in the Bergman span.

Bergman Coordinate Failure

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  You can map the polydisc with a rational mapping that mixes up the variables in such a way that the Jacobian, although rational, doesn't belong to the Bergman span.

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Recall that circular domains containing 0 are QD's; they're even QDP's. Among those are complete Reinhardt domains (which you can plot in $|z_j|$ coordinates.) You can make the boundary as bad as possible in certain directions.

Circular Domains

In several dimensions, circular domains containing the origin all have the property that their QI features only one point, 0. Bell asked whether the only other possible '1-point' QD's in several dimensions are images of circular domains under maps with constant Jacobian.

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The answer is no; you can map a polydisc via a mapping with linear Jacobian and end up in a QD whose QI has only one point (but possibly many terms).
If we further insist that the QI should have only one term, then the question is open.

Work to be done

Ongoing questions:

- Can all convex smooth domains be approximated by quadrature domains?
- Are all pseudoconvex domains biholomorphic to quadrature domains? (Is there a 'substitute Riemann mapping theorem' available?)
- In the plane I can show that nearby conformally equivalent quadrature domains can be transformed into one another with a homotopy where each intermediate shape is also a quadrature domain. Can you deform them that way in several variables as well?

Thanks!