WEAK AND STRONG ASYMPTOTICS FOR THE POLLACZEK MULTIPLE ORTHOGONAL POLYNOMIALS

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Multiple orthogonal polynomials?

MULTIPLE ORTHOGONAL POLYNOMIALS

Given two positive Borel measures μ_1 and μ_2 on $\mathbb{R}$, and a multi-index $\vec{n} = (n_1, n_2) \in \mathbb{N}^2$, the monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}| = n_1 + n_2$ is a type II multiple orthogonal polynomial if it satisfies

$$\int P_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2.$$

These polynomials appear as denominators of the simultaneous or Hermite–Padé approximants to a set of two functions: the Cauchy transform of μ_j's

$$\widehat{\mu}_j(z) = \int \frac{d\mu_j(x)}{z - x}$$

We seek polynomials $P_{\vec{n}}$ and $Q_{\vec{n},j}$, $j = 1, 2$, such that

$$R_{\vec{n},j}(z) = P_{\vec{n}}(z) \widehat{\mu}_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}\left(z^{-n_j - 1}\right), \quad z \to \infty$$

MULTIPLE ORTHOGONAL POLYNOMIALS

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These are $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of $P_{\vec{n}}$.

If this system has a unique solution, then we say that $\vec{n}$ is a normal index for type II. In this case,

$$\int P_{\vec{n}-\vec{e}_j}(x) x^{n_j - 1} d\mu_j(x) \neq 0, \quad j = 1, 2.$$

MULTIPLE ORTHOGONAL POLYNOMIALS

Existence?

Normality?

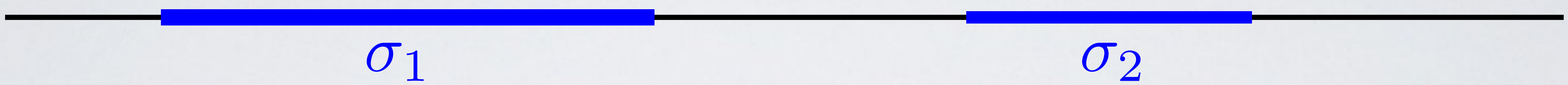
These are very difficult questions to address in the full generality.

There are easy to produce examples when the answer is **No**.

But there are classes of pairs of measures (μ_1, μ_2) for which the answer is **Yes**.

NIKISHIN SYSTEM

Start with two measures, σ_1 and σ_2, supported on $\mathbb{R}$, such that their supports are disjoint,

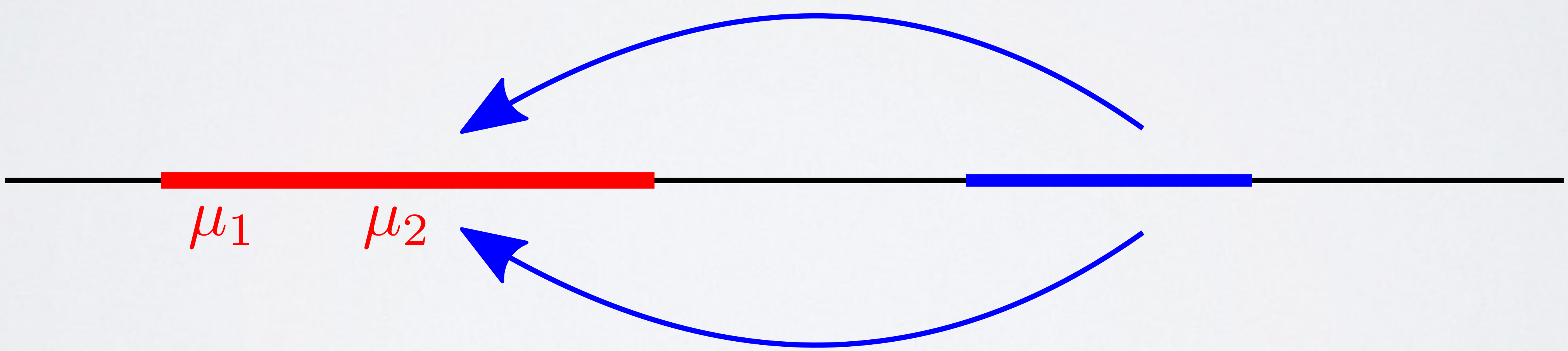


and define

$$d\mu_1 = d\sigma_1, \quad d\mu_2 = \widehat{\sigma_2}\, d\sigma_1$$

where (recall)

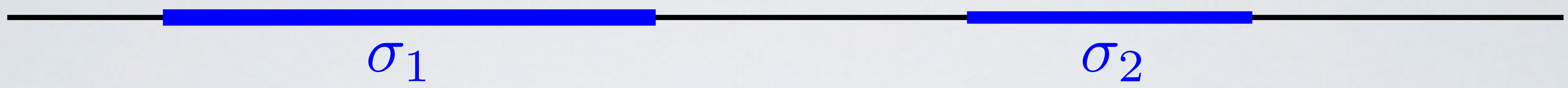
$$\widehat{\sigma}(z) = \int \frac{d\sigma(x)}{z - x}$$



(μ_1, μ_2) is a Nikishin system of measures.

For such class of measures, many multi-indices are normal (G. López Lagomasino).

NIKISHIN SYSTEM



A crucial fact is that we can work out an additional set of orthogonality conditions. For instance, there exists a monic polynomial q_{n_2} of degree n_2 and all its zeros on $\mathrm{supp}(\sigma_2)$ such that

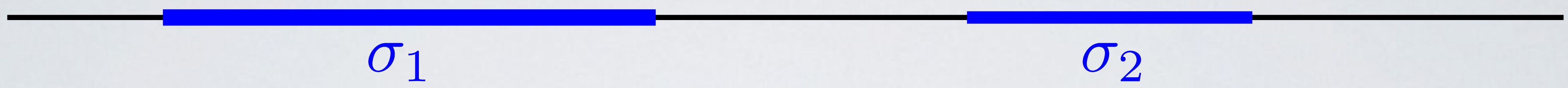
$$\int_{\mathrm{supp}(\sigma_1)} x^k P_{\vec{n}}(x) \frac{d\mu_1(x)}{q_{n_2}(x)} = 0, \quad k = 0, \ldots, |\vec{n}| - 1,$$

$$\int_{\mathrm{supp}(\sigma_2)} x^k q_{n_2}(x) \frac{f_{\vec{n}}(x) d\sigma_2(x)}{P_{\vec{n}}(x)} = 0, \quad k = 0, \ldots, n_2 - 1,$$

where $f_{\vec{n}}$ is the Cauchy transform of a certain explicit function.

So, we can "trade" two orthogonality conditions on the same interval for a full set of orthogonality conditions on disjoint intervals, but with respect to varying measures.

NIKISHIN SYSTEM



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where $f_{\vec{n}}$ is the Cauchy transform of a certain explicit function.

The asymptotic zero distribution of the multiple orthogonal polynomials $P_{\vec{n}}$ is described in terms of an associated vector equilibrium problem.

NIKISHIN SYSTEM

Recall that for a measure μ we define its logarithmic potential

$$V^{\mu}(z) = \int \log \frac{1}{|z-x|}\, d\mu(x)$$

Given a polynomial P, we also define its normalized zero-counting measure

$$\nu_P := \frac{1}{\deg P} \sum_{P(x)=0} \delta_x$$

in such a way that

$$\frac{1}{\deg P} \log |P(z)| = -V^{\nu_P}(z).$$

NIKISHIN SYSTEM

The orthogonality conditions

$$\int_{\mathrm{supp}(\sigma_1)} x^k P_{\vec{n}}(x) \frac{d\mu_1(x)}{q_{n_2}(x)} = 0, \quad k = 0, \ldots, |\vec{n}| - 1,$$

$$\int_{\mathrm{supp}(\sigma_2)} x^k q_{n_2}(x) \frac{f_{\vec{n}}(x) d\sigma_2(x)}{P_{\vec{n}}(x)} = 0, \quad k = 0, \ldots, n_2 - 1,$$

imply that, in the simplest case, and assuming that $n_1 = n_2 = n$,

$$\lim_n \nu_{P_{\vec{n}}} = \lambda_1/2, \quad \lim_n \nu_{q_n} = \lambda_2,$$

where $|\lambda_1| = 2$, $|\lambda_2| = 1$, $\mathrm{supp}(\lambda_j) \subset \mathrm{supp}(\sigma_j)$, $j = 1, 2$, and

$$\boxed{2V^{\lambda_i}(x) - V^{\lambda_j}(x) \begin{cases} = \omega_i = \mathrm{const}, & x \in \mathrm{supp}(\lambda_i), \\ \geq \omega_i, & x \in \mathrm{supp}(\sigma_i) \setminus \mathrm{supp}(\lambda_i), \end{cases}}$$

$i, j \in \{1, 2\}$, $i \neq j$.

Multiple Pollaczek polynomials

MULTIPLE POLLACZEK POLYNOMIALS

We have two absolutely continuous measures $d\mu_j(x) = w_j(x)dx$ on $\mathbb{R}_+ = [0, +\infty)$ given by

$$w_1(x) = \frac{dx}{\sinh \frac{\pi\sqrt{x}}{2}}, \quad w_2(x) = \frac{1}{\cosh \frac{\pi\sqrt{x}}{2}} \frac{dx}{\sqrt{x}} = \frac{\tanh \frac{\pi\sqrt{x}}{2}}{\sqrt{x}} w_1(x)$$

Decomposing $\tanh(\pi z/2)/z$ into simple fractions, we get

$$\frac{\tanh \frac{\pi\sqrt{z}}{2}}{\sqrt{z}} = \int \frac{d\sigma_2(x)}{z-x}$$

where

$$\sigma_2 = \frac{4}{\pi} \sum_{k\in\mathbb{Z}_+} \delta_{-(2k+1)^2}$$

Thus, (μ_1, μ_2) form a Nikishin system, BUT (i) $\mathrm{supp}(\mu_j)$ is unbounded and (ii) σ_2 is discrete.

We can expect two additional ingredients in the vector equilibrium problem: an external field (coming from the need of rescaling) and an upper constraint (originated by the discrete orthogonality).

Weak asymptotics

EQUILIBRIUM PROBLEM

We consider the diagonal sequence $n_1 = n_2 = n$, and the rescaled polynomials

$$Q_n(x) = c_n P_{\vec{n}}(4n^2 x) = x^{2n} + \text{lower degree terms}, \quad c_n = (4n^2)^{-2n}$$

Taking into account that

$$\varphi(x) = \lim_n \frac{1}{2n} \log\left(|\sinh(n\pi\sqrt{x})|^{1/2} \right) = \frac{\pi}{4}\sqrt{x}$$

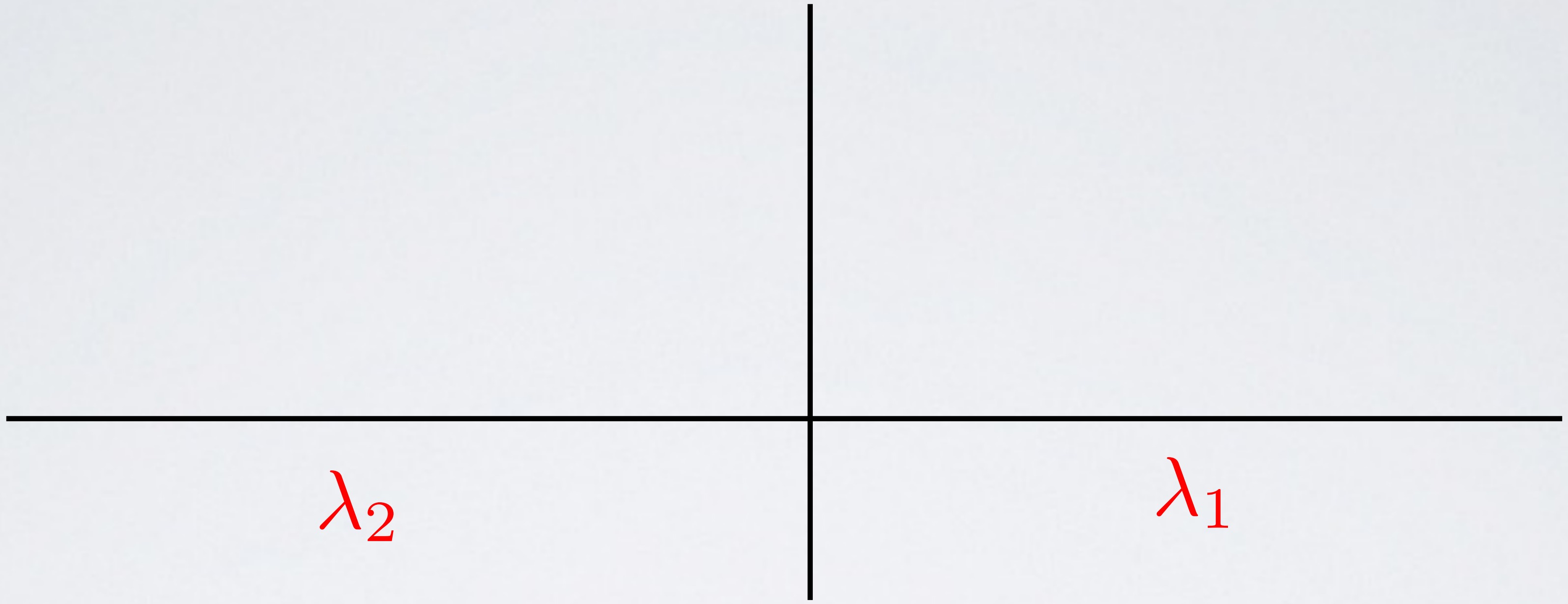
and that for $T < 0$,

$$\lim_n \int_{[T,0]} \frac{1}{2n} d\sigma_{2,n}(t) = \lim_{n \in 2\mathbb{N}} \frac{\sharp\{k : (2k+1)^2 \leq 4n^2|T|\}}{n} = \int_{[T,0]} \frac{|dt|}{2\sqrt{|t|}}$$

we arrive at the following equilibrium problem for the weak-* limit of the zeros of Q_n.

EQUILIBRIUM PROBLEM

λ_1, λ_2: $|\lambda_1| = 2$, $\mathrm{supp}(\lambda_1) \subset \mathbb{R}_+$, and $|\lambda_2| = 1$, $\mathrm{supp}(\lambda_2) \subset \mathbb{R}_-$,



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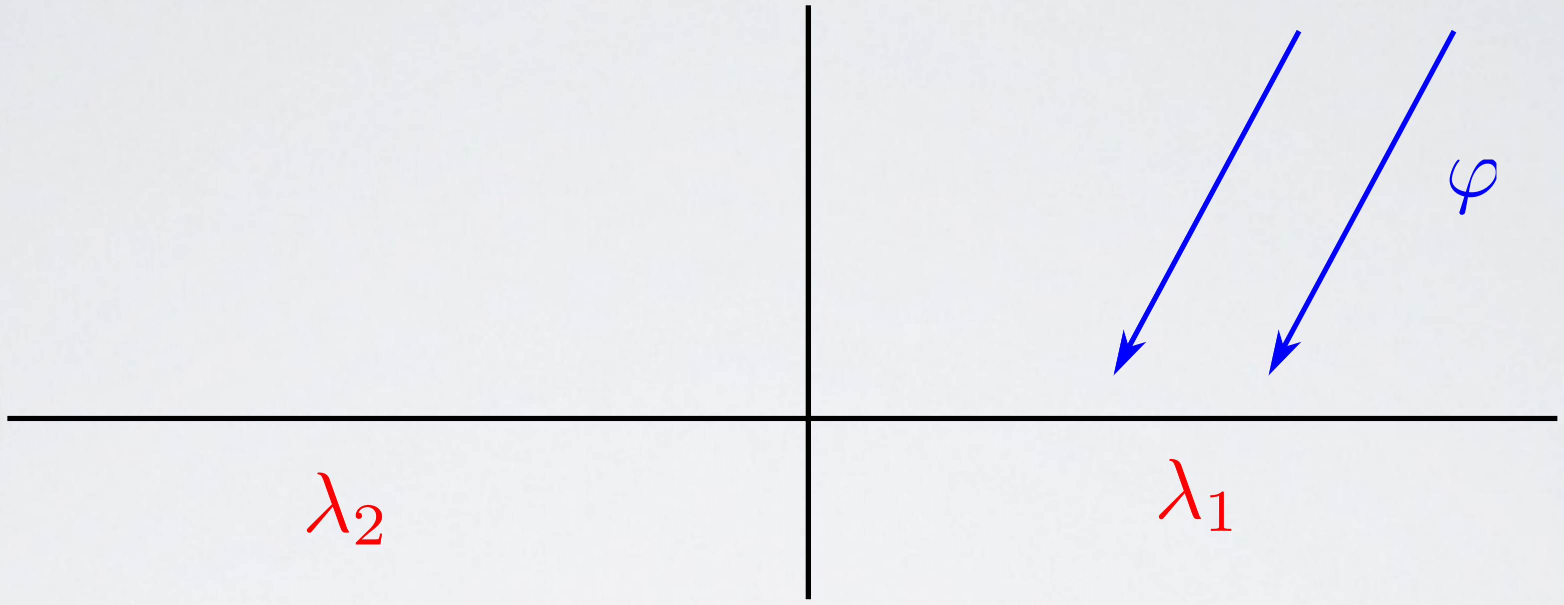

such that with $\varphi(x) = \pi\sqrt{x} > 0$ $\mathbb{R}_+$,

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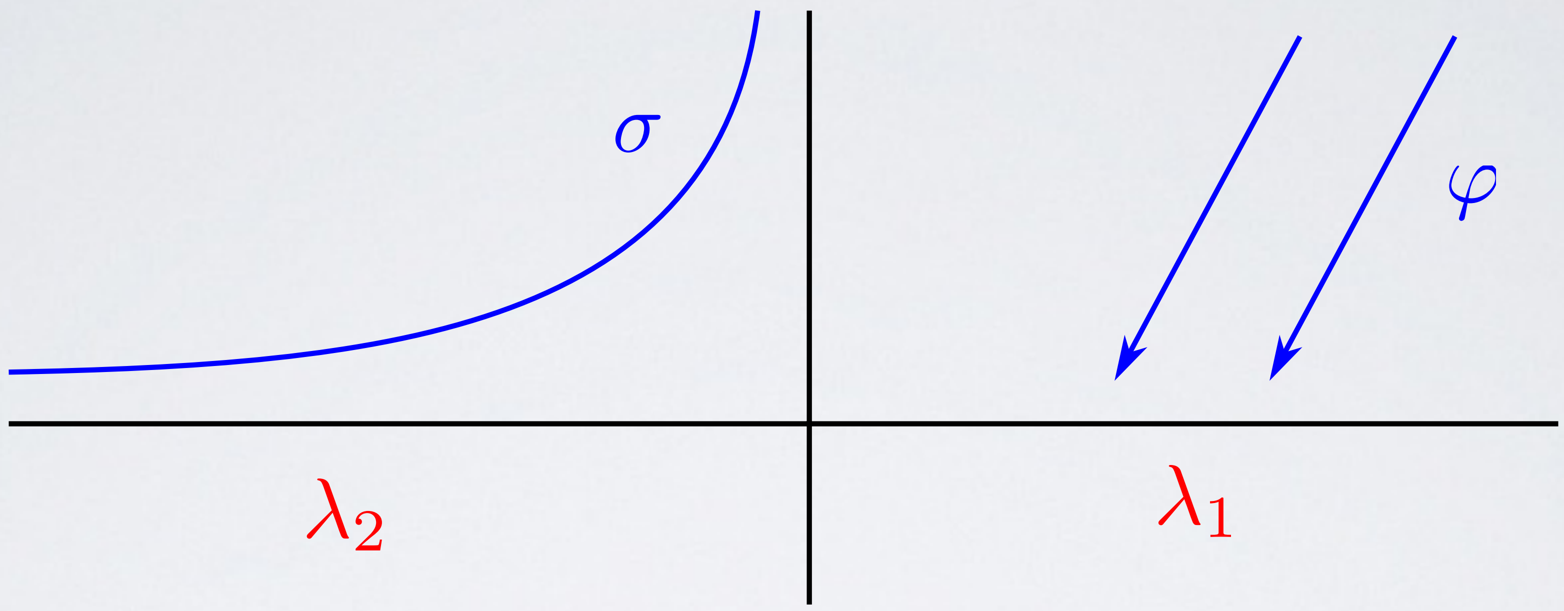

such that with $\varphi(x) = \pi\sqrt{x} > 0$ $\mathbb{R}_+$, and $d\sigma(x) = \dfrac{dx}{2\sqrt{|x|}}$, $x \in \overline{\mathbb{R}}_-$,

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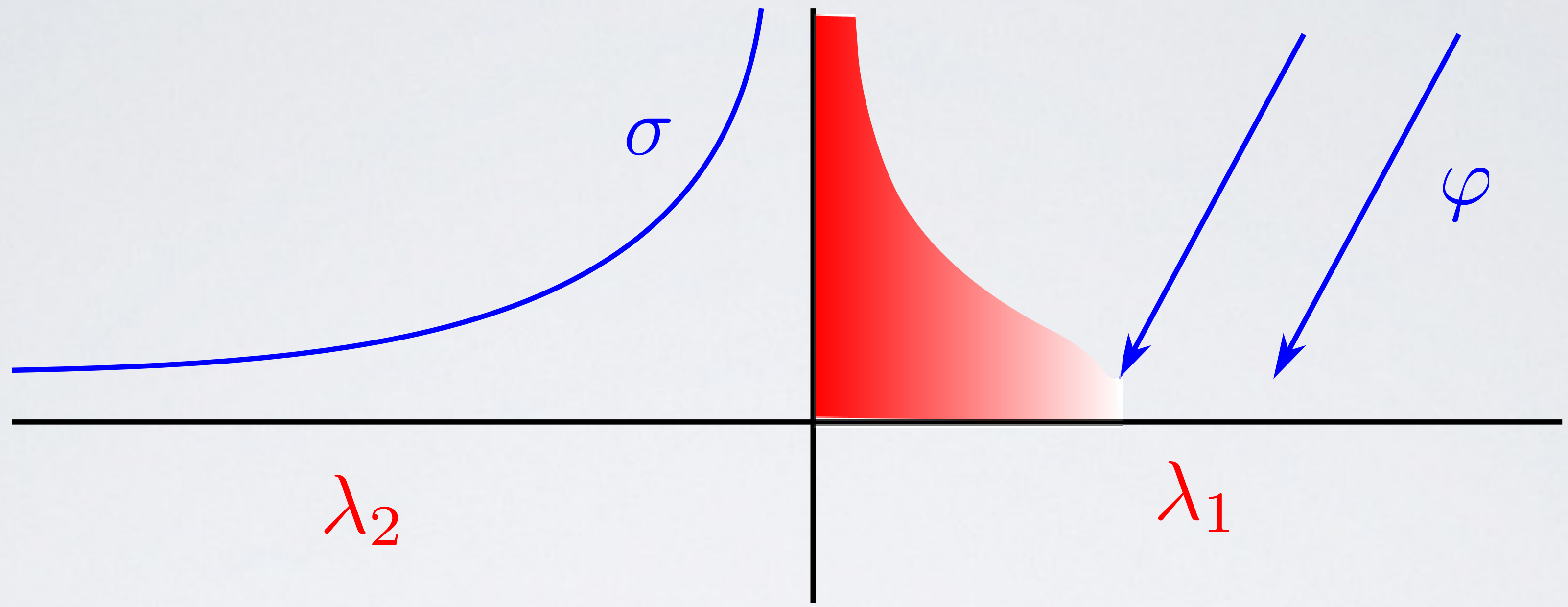

such that with $\varphi(x) = \pi\sqrt{x} > 0$ $\mathbb{R}_+$, and $d\sigma(x) = \dfrac{dx}{2\sqrt{|x|}}$, $x \in \overline{\mathbb{R}}_-$,

$$2V^{\lambda_1}(x) - V^{\lambda_2}(x) + \varphi(x) \begin{cases} = \omega, & x \in [0, p_+], \\ > \omega, & x > p_+; \end{cases} \qquad p_+ = \left(\frac{2}{\sqrt{5}-1}\right)^5$$

EQUILIBRIUM PROBLEM

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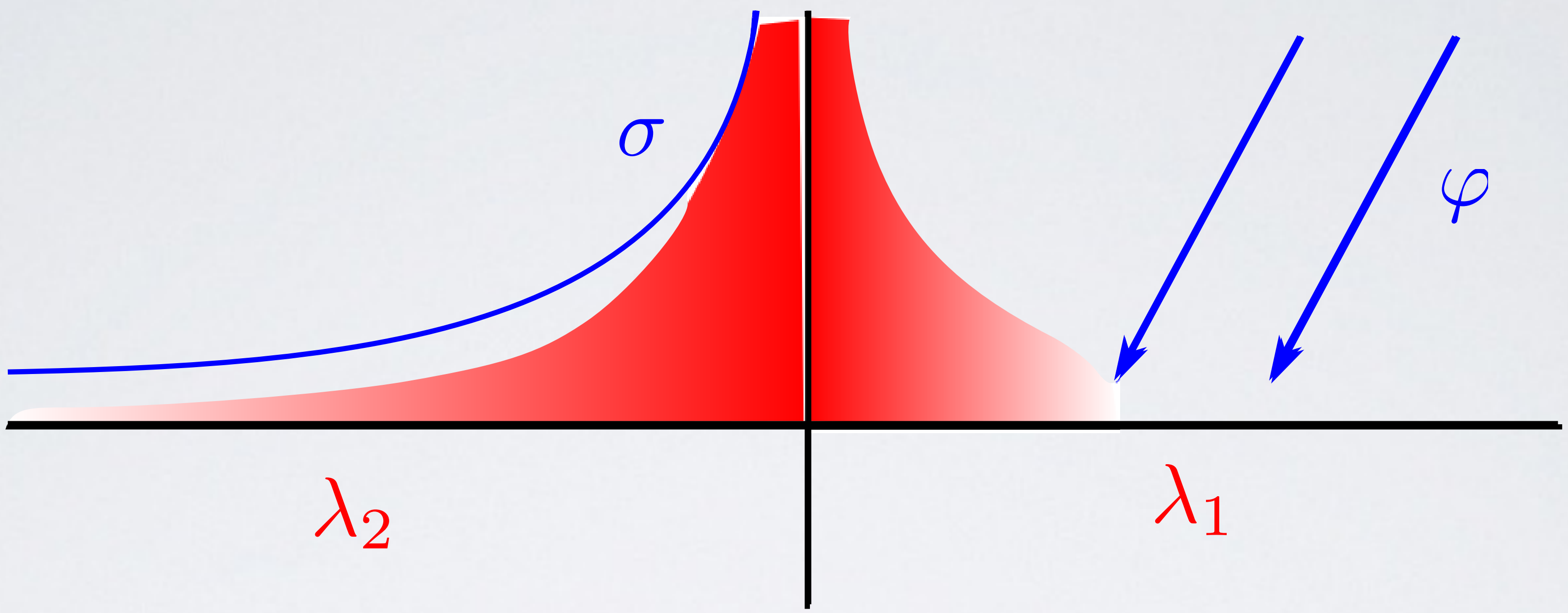

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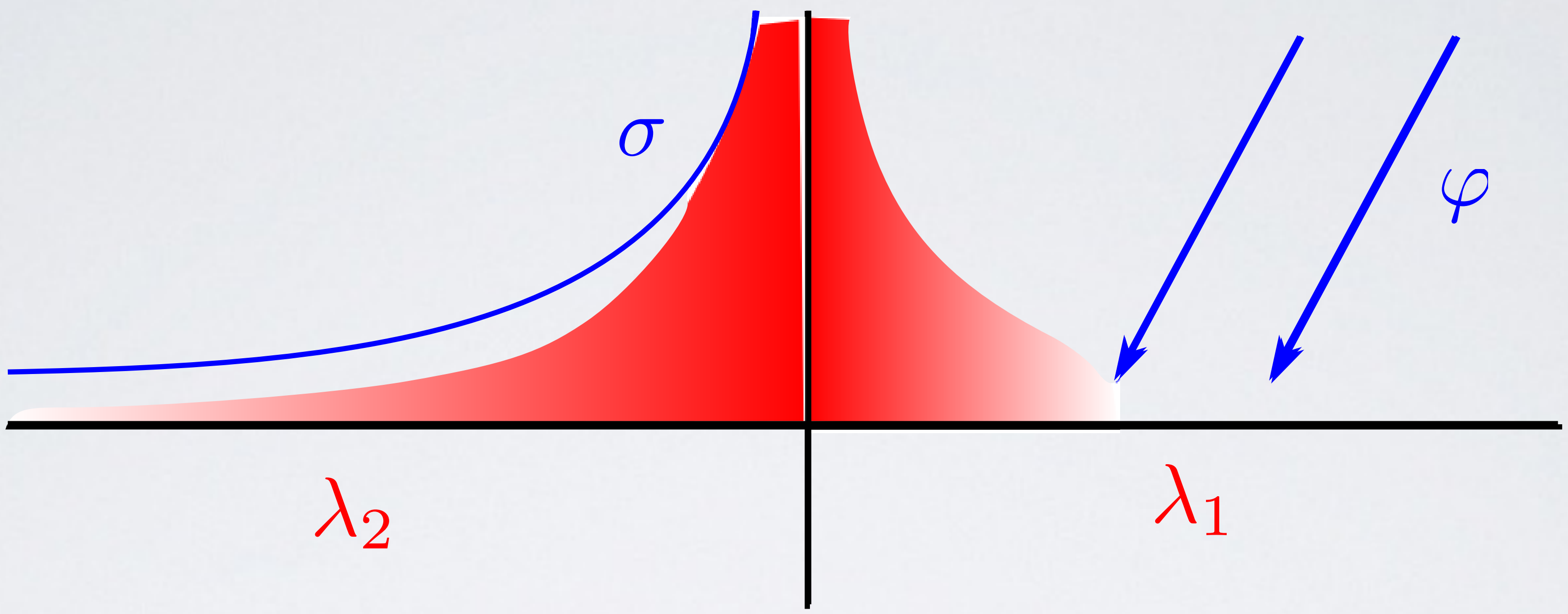
$$2V^{\lambda_2}(x) - V^{\lambda_1}(x) \begin{cases} = 0, & x \le p_-, \\ < 0, & x \in (p_-, 0), \end{cases} \qquad p_- = -\left(\frac{\sqrt{5}-1}{2}\right)^5$$

and $\lambda_2 \le \sigma$.

WEAK ASYMPTOTICS

λ_1, λ_2: $|\lambda_1| = 2$, $\mathrm{supp}(\lambda_1) \subset \mathbb{R}_+$, and $|\lambda_2| = 1$, $\mathrm{supp}(\lambda_2) \subset \mathbb{R}_-$,



Then,

$$\lim_n \nu_{Q_n} = \lambda_1/2$$

In particular,

$$\boxed{\lim_n |Q_n(z)|^{1/(2n)} = \exp\left(-\frac{1}{2} V^{\lambda_1}(z)\right), \quad z \in \mathbb{C} \setminus [0, p_+]}$$

This is a weak exterior asymptotics for Q_n's.

What about
the strong
asymptotics?

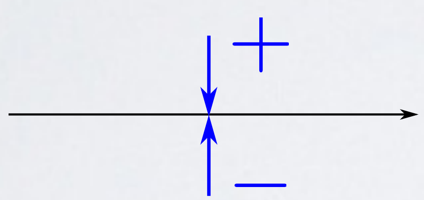
RIEMANN-HILBERT CHARACTERIZATION

Strong asymptotics is based on the following RHP characterization of Q_n's:

Find $\boldsymbol{Y}(z) \in \mathbb{C}^{3\times 3}$ analytic in $\mathbb{C} \setminus \mathbb{R}_+$ such that

- on $\mathbb{R}_+$,

$$\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x) \begin{pmatrix} 1 & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where 

- As $z \to \infty$, $z \in \mathbb{C} \setminus \mathbb{R}_-$,

$$\boldsymbol{Y}(z) = \left(\boldsymbol{I} + \mathcal{O}(z^{-1})\right) \operatorname{diag}\left(z^{2n}, z^{-n}, z^{-n}\right)$$

- $\boldsymbol{Y}(z) = \mathcal{O}\left(\; 1 \;\middle|\; |z|^{-1/2} \;\middle|\; |z|^{-1/2} \; \right)$, $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$.

Then $\boldsymbol{Y}_{11} = Q_n$!

THE R-H STEEPEST DESCENT ANALYSIS

Idea of the asymptotic analysis:

$$\text{Start with } \mathbf{Y} \Rightarrow \text{ get an asymptotic expression for } \mathbf{Y}$$

$$\downarrow$$

$$\mathbf{T}$$

$$\downarrow$$

$$\mathbf{U}$$

$$\downarrow$$

conclude that

$$\mathbf{I} + \text{ small} = \mathbf{S} \text{ such that } \begin{cases} \mathbf{S}(\infty) = \mathbf{I} \\ \mathbf{S}_+(z) = \mathbf{S}_-(z)\,(\mathbf{I} + \text{ small}) \end{cases}$$

This is just a roadmap.

Along this path, the equilibrium measures λ_1, λ_2 should appear, but not only…

THE R-H STEEPEST DESCENT ANALYSIS

Step 1: create 2×2 blocks in the jump matrix

$$\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x) \begin{pmatrix} 1 & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Standard procedure for the Nikishin system: use the ratio

$$\frac{w_{2,n}}{w_{1,n}} = \widehat{\sigma}_2$$

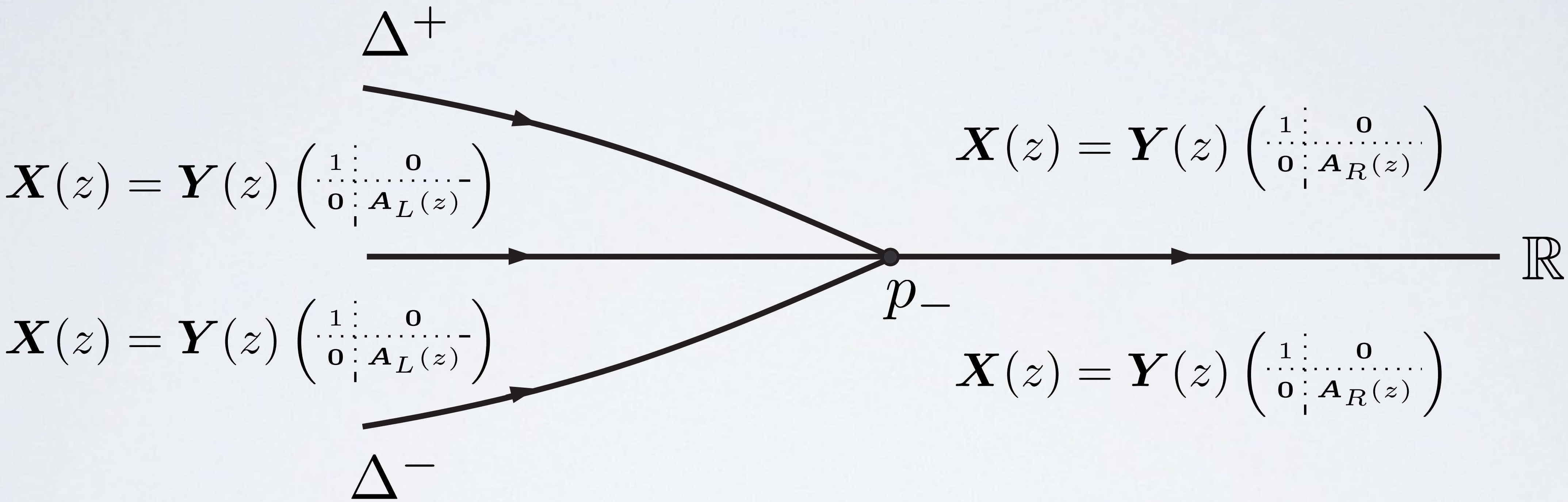
But in our case this function is meromorphic with poles on $\mathbb{R}_-$. Bad news!

Instead, we use the specific properties of the weights $w_{j,n}$.

THE R-H STEEPEST DESCENT ANALYSIS

Define $\color{red}{\upsilon} = \upsilon(z) := \exp\left(\pi z^{1/2}\right)$, $z \in \mathbb{C} \setminus \mathbb{R}_-$, and

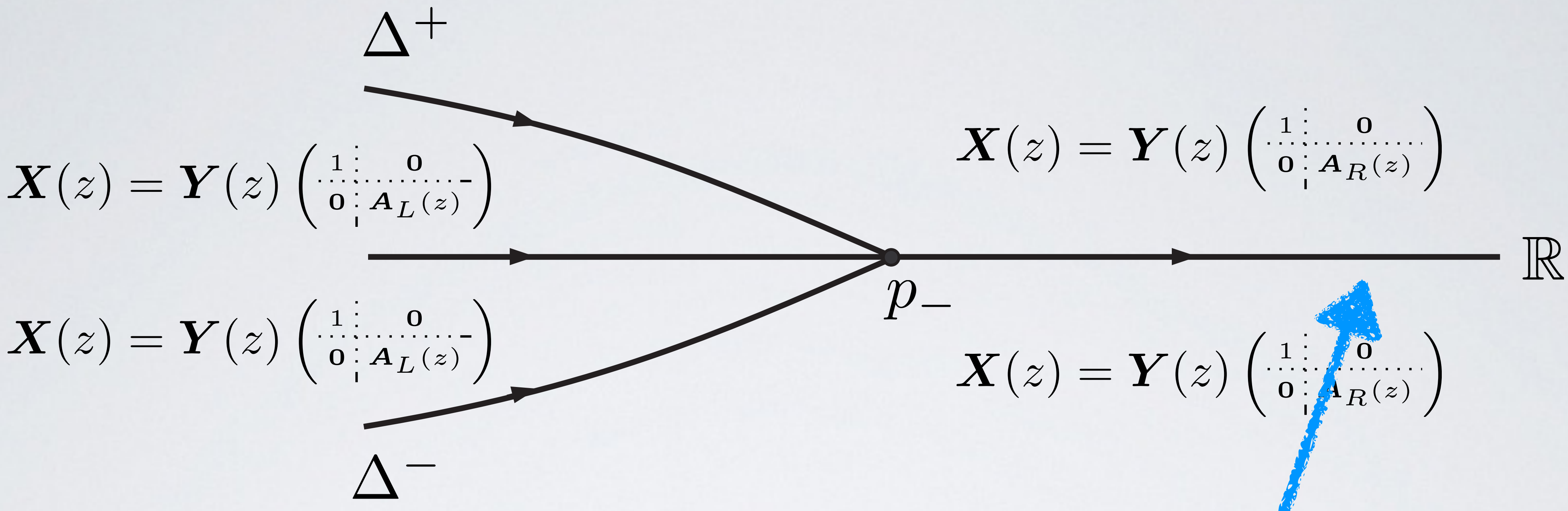
$$\color{red}{\boldsymbol{A}_L}(z) := \begin{pmatrix} 1 & -1/(2z^{1/2}) \\ z^{1/2} & 1/2 \end{pmatrix}, \quad \color{red}{\boldsymbol{A}_R}(z) = \begin{pmatrix} 1 & -\frac{1}{z^{1/2}\upsilon^n w_1} \\ z^{1/2} & \frac{1}{z^{1/2}\upsilon^n w_2} \end{pmatrix}$$



Then, $\color{red}{\boldsymbol{X}} = \boldsymbol{X}_n$ is holomorphic in $\mathbb{C} \setminus (\mathbb{R} \cup \Delta^+ \cup \Delta^-)$, and

$$\boldsymbol{X}_+(z) = \boldsymbol{X}_-(z)\color{red}{\boldsymbol{J}_{\boldsymbol{X}}}(z)$$

THE R-H STEEPEST DESCENT ANALYSIS



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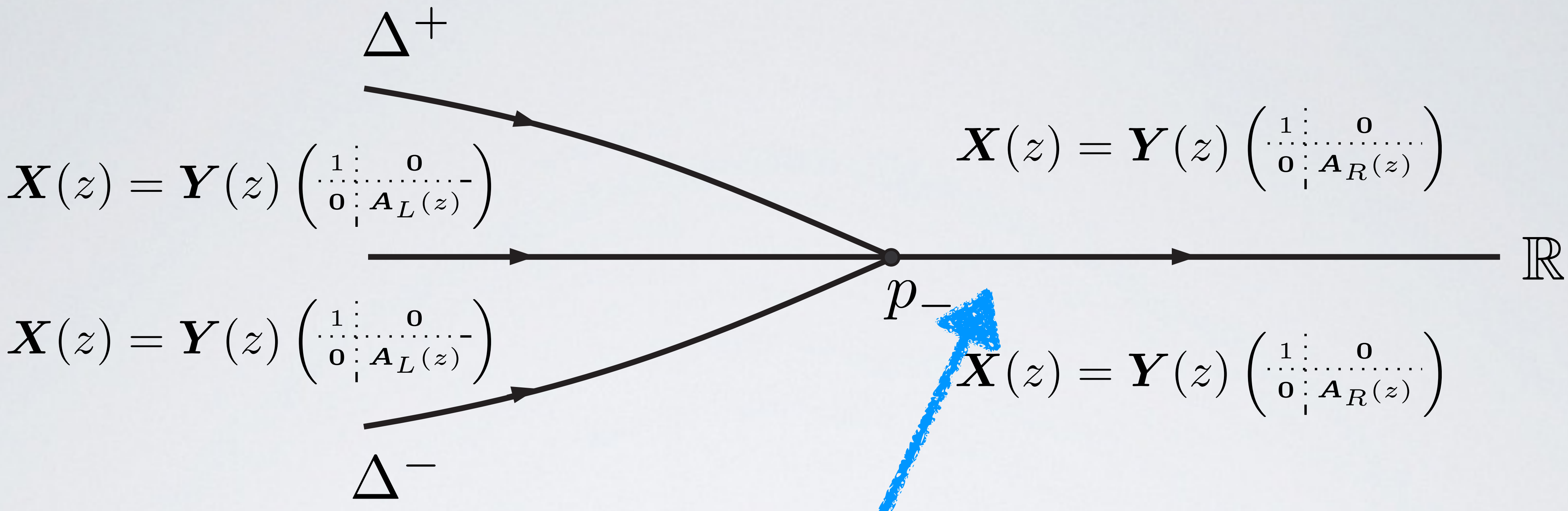
$$\boldsymbol{X}_+(z) = \boldsymbol{X}_-(z)\boldsymbol{J}_{\boldsymbol{X}}(z)$$

where

$$\boldsymbol{J}_{\boldsymbol{X}}(z) = \begin{pmatrix} 1 & \frac{4\upsilon^n}{\upsilon^{2n} - \upsilon^{-2n}} & 0 \\ & 1 & \\ & & 1 \end{pmatrix}, \quad x > 0$$

THE R-H STEEPEST DESCENT ANALYSIS



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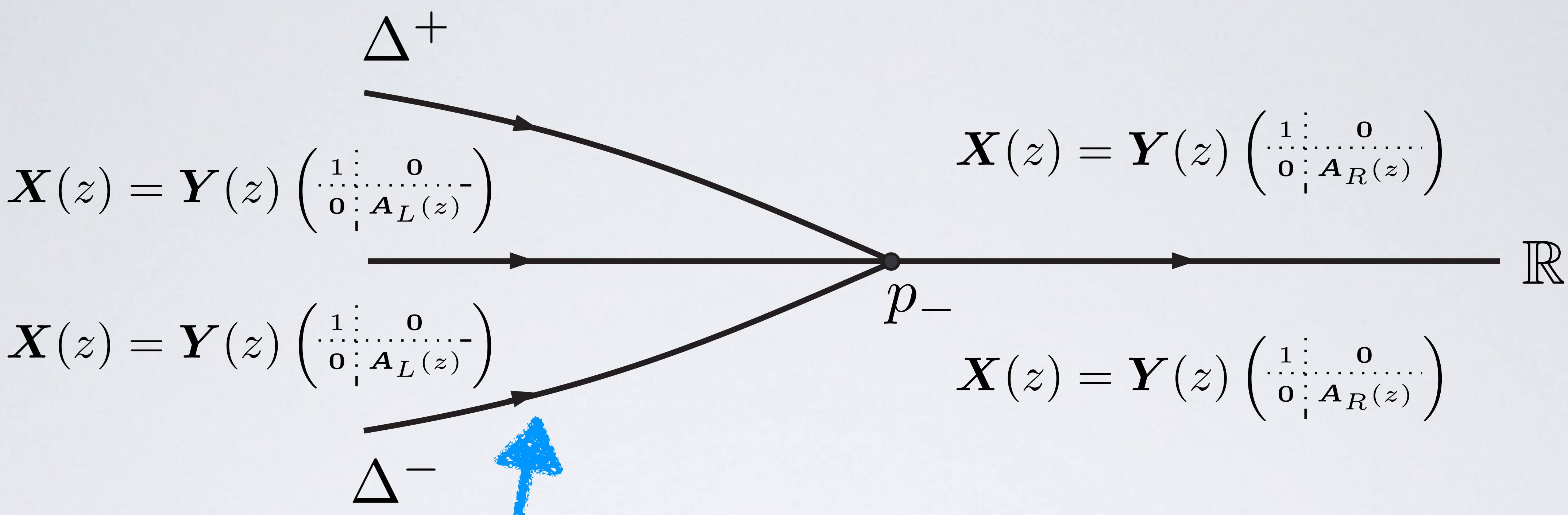
$$\boldsymbol{X}_+(z) = \boldsymbol{X}_-(z)\boldsymbol{J}_{\boldsymbol{X}}(z)$$

where

$$\boldsymbol{J}_{\boldsymbol{X}}(z) = \begin{pmatrix} 1 & & \\ & v_+^{2n} & \\ & 2x_+^{1/2} & v_+^{-2n} \end{pmatrix}, \quad x \in (p_-, 0)$$

THE R-H STEEPEST DESCENT ANALYSIS



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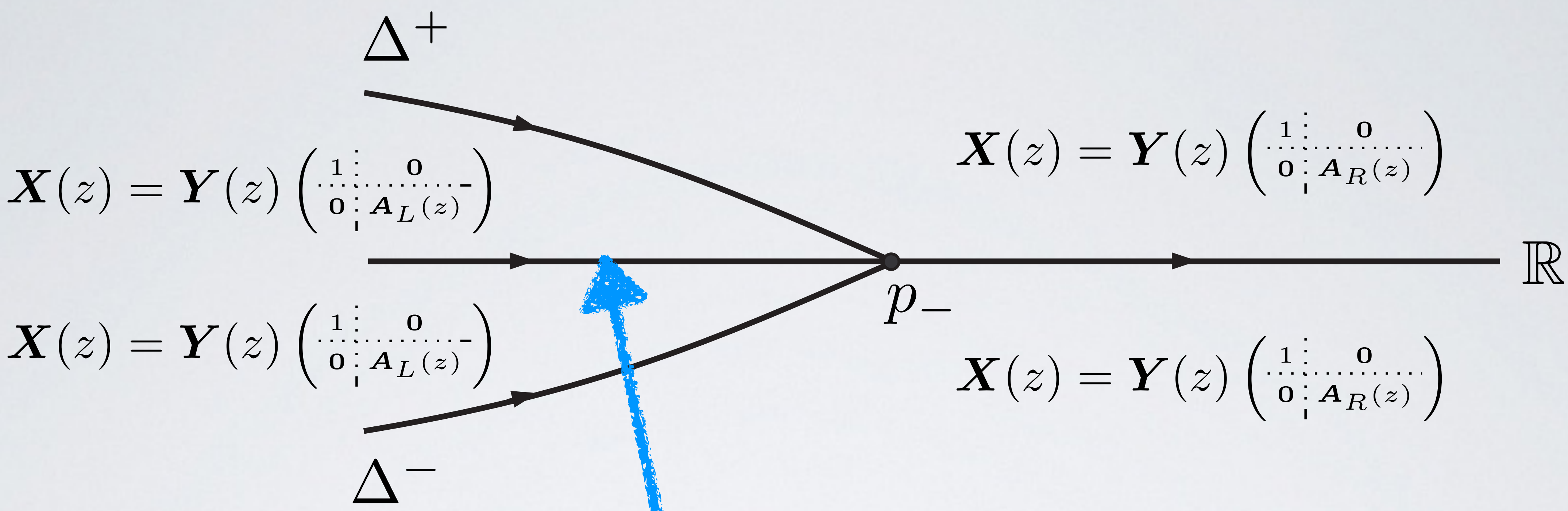
where

$$\boldsymbol{J}_{\boldsymbol{X}}(z) = \left(\begin{array}{c:c} 1 & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & \boldsymbol{B}^{\pm 1}(z) \end{array}\right), \quad z \in \Delta^{\pm}, \quad \boldsymbol{B}(z) := \begin{pmatrix} 1 & \frac{1}{2z^{1/2}\upsilon^{2n}} \\ & 1 \end{pmatrix}$$

THE R-H STEEPEST DESCENT ANALYSIS



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where

$$\boldsymbol{J}_{\boldsymbol{X}}(z) = \begin{pmatrix} 1 & & \\ & 0 & -1/(2x_+^{1/2}) \\ & 2x_+^{1/2} & 0 \end{pmatrix}, \quad x \in (-\infty, p_-)$$

THE R-H STEEPEST DESCENT ANALYSIS

Step 2: normalization at infinity.

Here we use our equilibrium measures λ_1 and λ_2!

We define the g-functions

$$g_j(z) = \int \log(z-t)\, d\lambda_j(t), \quad j = 1, 2$$

and take

$$\boldsymbol{U}(z) = \text{const}\, \boldsymbol{X}(z)\, \text{diag}\left(e^{-n(g_1(z)+\omega)}, e^{n(g_1(z)-g_2(z))}, e^{ng_2(z)}\right)$$

ω a constant (the "equilibrium constant").

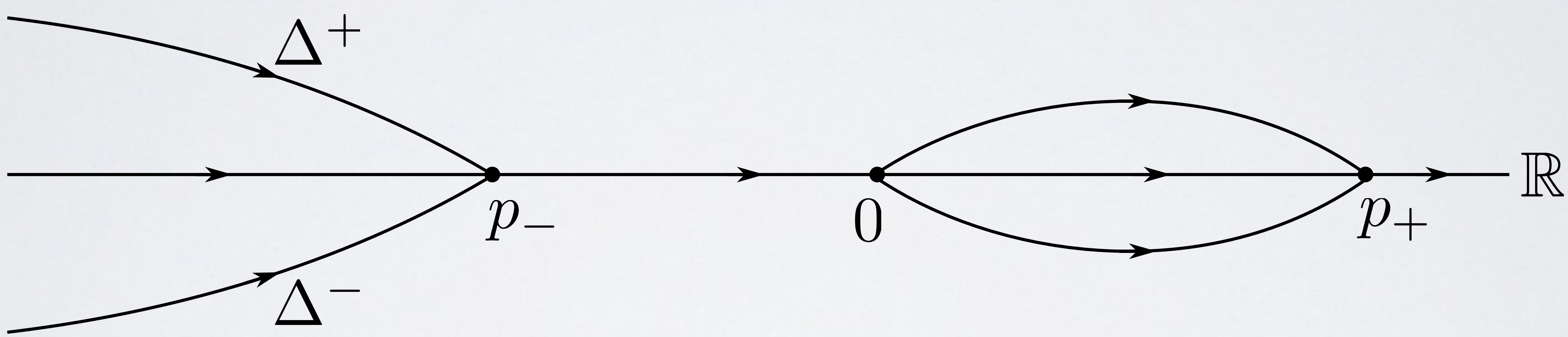
This transformation gives us jumps of $\boldsymbol{U}$ that are uniformly close to $\boldsymbol{I}$, except at $(-\infty, p_-]$ and $[0, p_+]$.

THE R-H STEEPEST DESCENT ANALYSIS

Step 3: lens opening.

We need an additional transformation in order to "peel off" all the irrelevant jumps.

We factor the jump on $[0, p_+]$ and replace it by three consecutive jumps (two on the newly introduced curves):



At this point, let us ignore all the jumps that are uniformly close to $\boldsymbol{I}$. We are left with a model problem that should capture the main features of the current RH problem.

This is what is usually called the global parametrix for our problem.

It will be used to kill the last non-trivial jumps and get to our dream: to have all jumps "almost $\boldsymbol{I}$".

THE R-H STEEPEST DESCENT ANALYSIS

Step 4: global parametrix.

We seek $\boldsymbol{N}$, holomorphic in $\mathbb{C} \setminus ((-\infty, p_-] \cup [0, p_+])$, such that

$$\boldsymbol{N}_+(x) = \boldsymbol{N}_-(x) \begin{pmatrix} 0 & 4 & 0 \\ -1/4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (0, p_+),$$

$$\boldsymbol{N}_+(x) = \boldsymbol{N}_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/(2x_+^{1/2}) \\ 0 & 2x_+^{1/2} & 0 \end{pmatrix}, \quad x \in (-\infty, p_-),$$

$$\boldsymbol{N}(z) = \left(\boldsymbol{I} + \mathcal{O}(z^{-1})\right) \left(\begin{array}{c:c} 1 & \mathbf{0} \\ \hdashline \mathbf{0} & \boldsymbol{A}_L(z) \end{array}\right) \quad \text{as} \quad z \to \infty, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

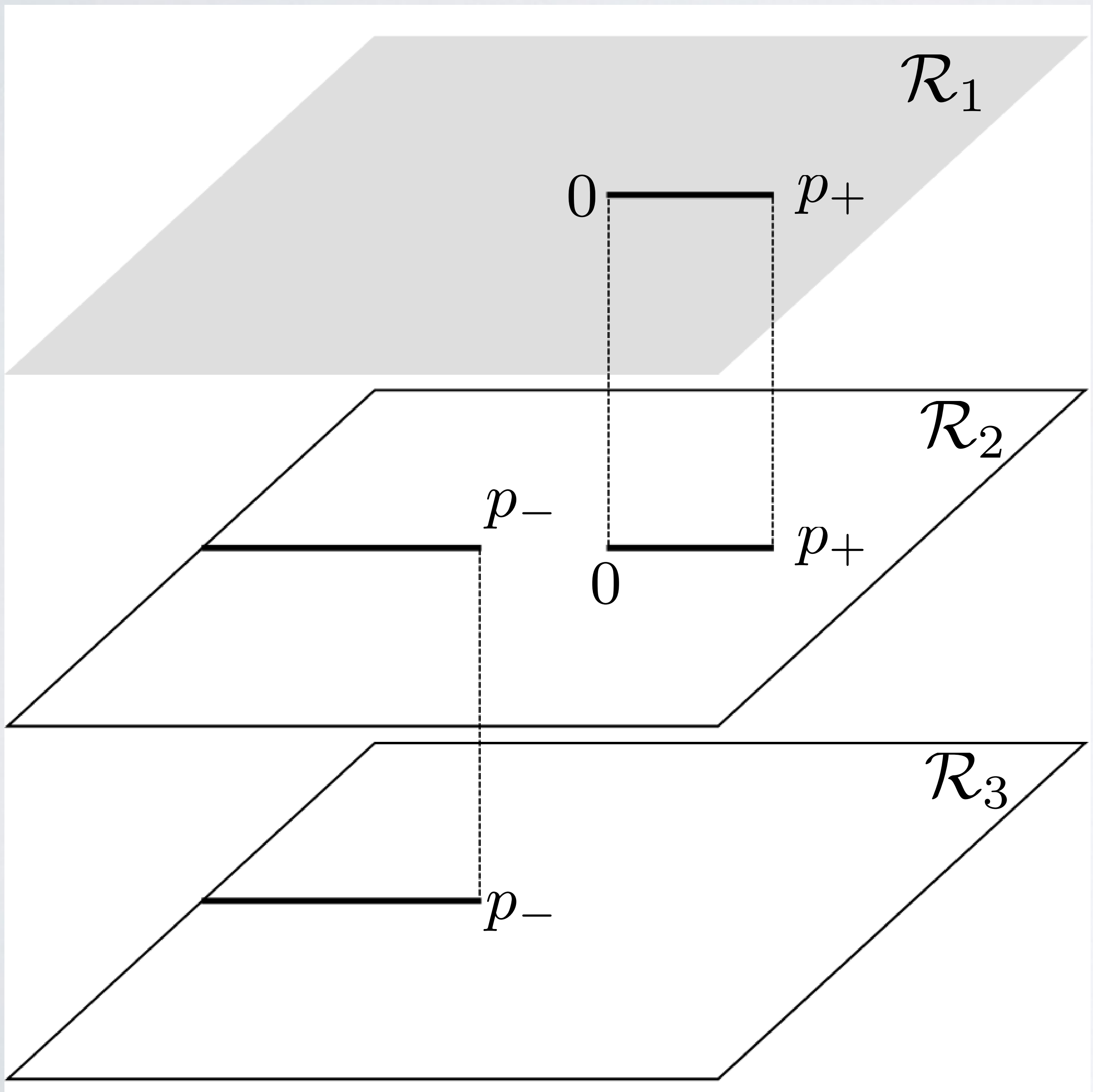
where

$$\boldsymbol{A}_L(z) = \begin{pmatrix} 1 & -1/(2z^{1/2}) \\ z^{1/2} & 1/2 \end{pmatrix}$$

THE R-H STEEPEST DESCENT ANALYSIS

Step 4: global parametrix.

This RHP is solved using the Riemann surface $\mathcal{R}$ constructed gluing the three copies of $\mathbb{C}$, as this:



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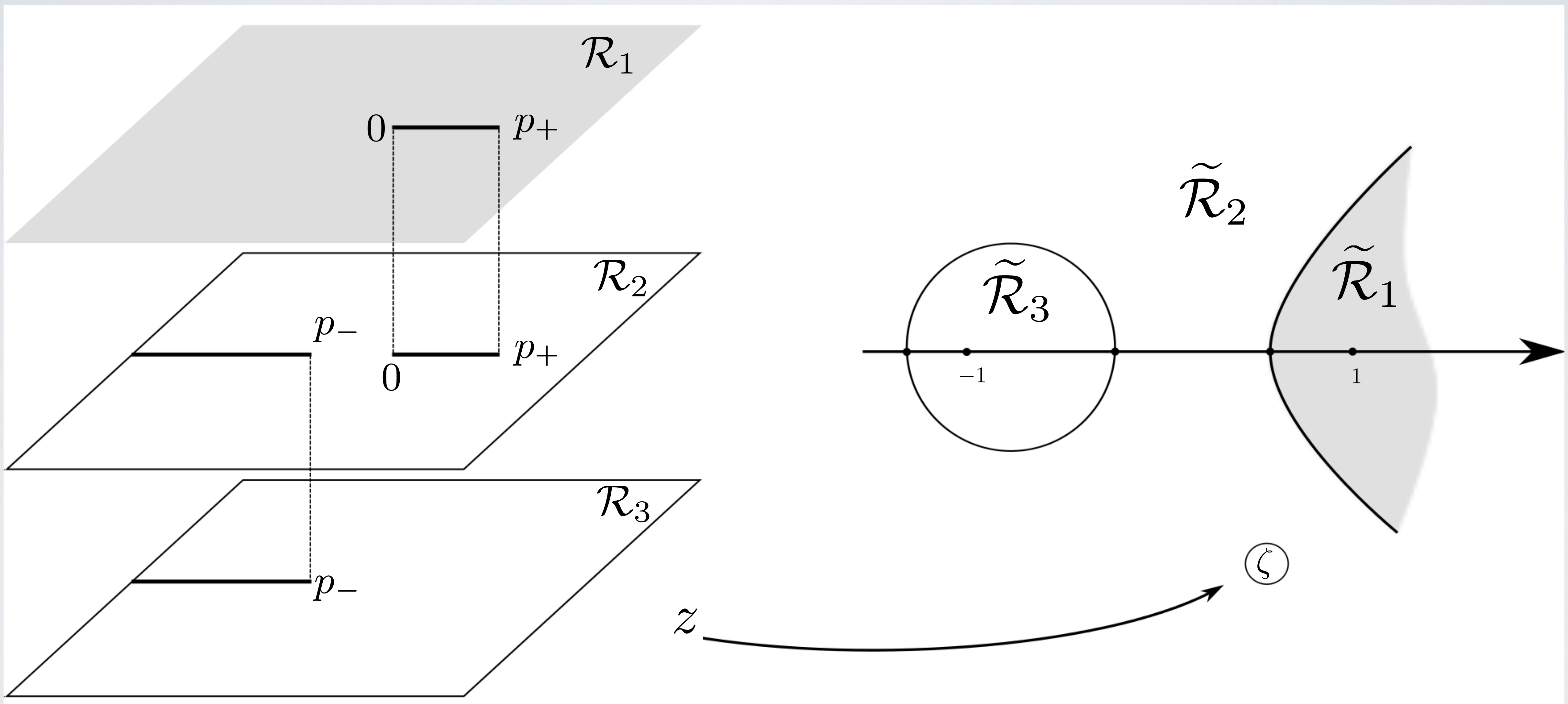
$$\boldsymbol{N}(z) = \left(\boldsymbol{I} + \mathcal{O}(z^{-1})\right) \left(\begin{array}{c|c} 1 & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{A}_L(z) \end{array}\right) \quad \text{as} \quad z \to \infty, \quad z \in \mathbb{C} \backslash \mathbb{R},$$

THE R-H STEEPEST DESCENT ANALYSIS

We use the explicit conformal mapping between $\mathcal{R}$ and the ζ-Riemann sphere

$$z = \frac{1+\zeta}{\zeta^2(1-\zeta)}$$



THE R-H STEEPEST DESCENT ANALYSIS

We use the explicit conformal mapping between $\mathcal{R}$ and the ζ-Riemann sphere

$$z = \frac{1+\zeta}{\zeta^2(1-\zeta)}$$

There are three inverse functions to $z = z(\zeta)$,

$$\zeta_1(z) = 1 - \frac{2}{z} - \frac{6}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right),$$

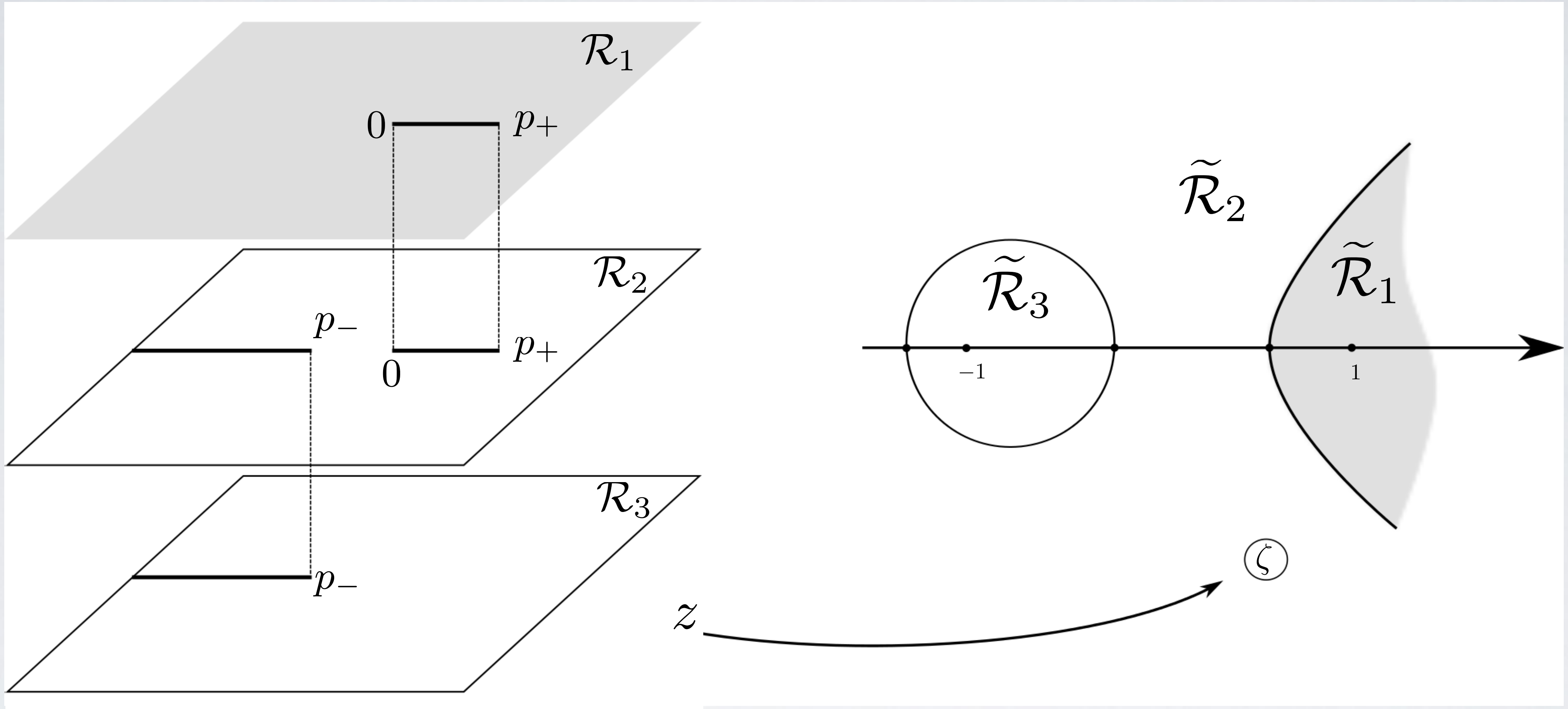
$$\zeta_2(z) = \frac{1}{z^{1/2}} + \frac{1}{z} + \frac{3}{2z^{3/2}} + \frac{3}{z^2} + \frac{55}{8z^{5/2}} + \mathcal{O}\left(\frac{1}{z^3}\right),$$

$$\zeta_3(z) = -\frac{1}{z^{1/2}} + \frac{1}{z} - \frac{3}{2z^{3/2}} + \frac{3}{z^2} - \frac{55}{8z^{5/2}} + \mathcal{O}\left(\frac{1}{z^3}\right)$$

as $z \to \infty$.

After some manipulations we can simplify the jumps even further, getting to a new matrix, $\widehat{\boldsymbol{N}}\ldots$

THE R-H STEEPEST DESCENT ANALYSIS

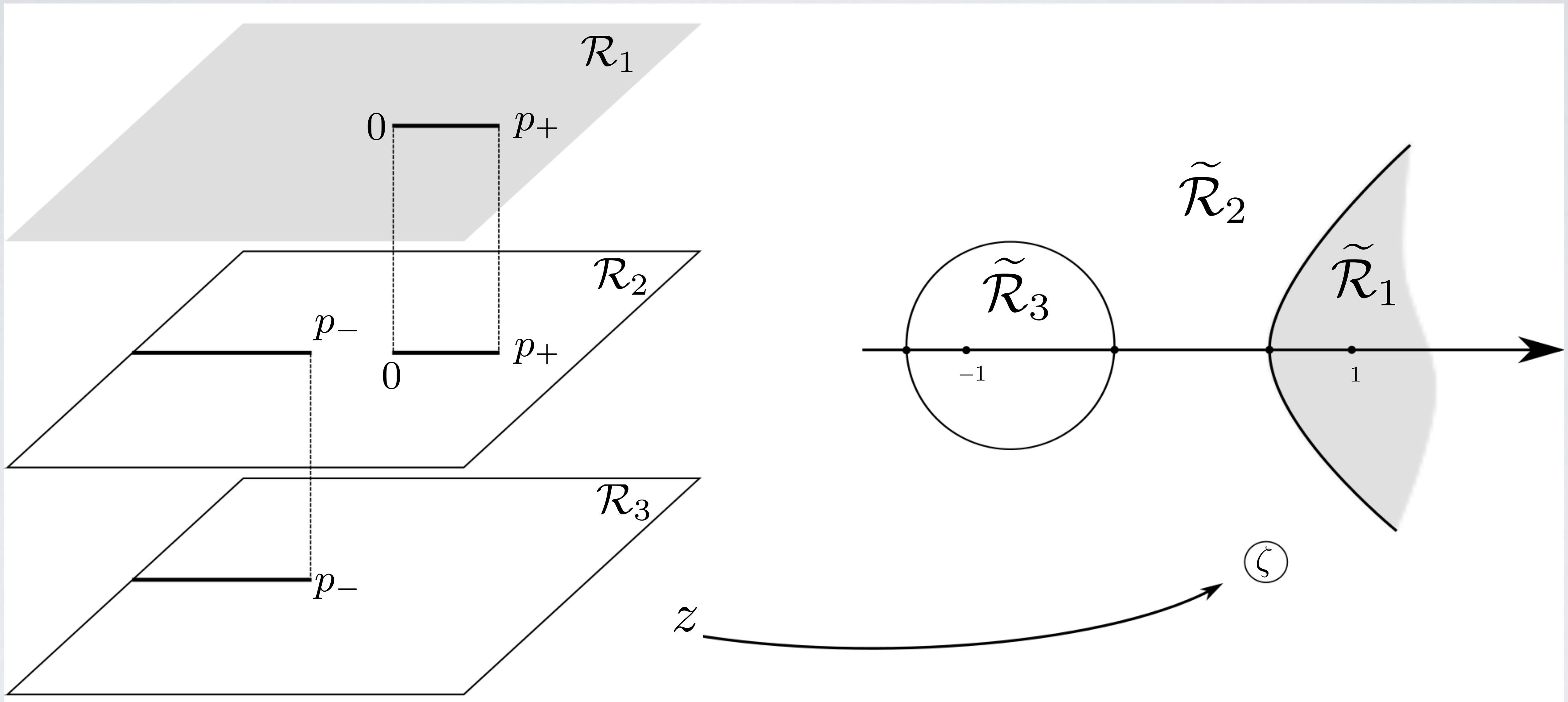


$$\widehat{\boldsymbol{N}}_+(z) = \widehat{\boldsymbol{N}}_-(z) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in (0, p_+),$$

$$\widehat{\boldsymbol{N}}_+(z) = \widehat{\boldsymbol{N}}_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad z \in (-\infty, p_-),$$

and a prescribed behavior as $z \to \infty$.

THE R-H STEEPEST DESCENT ANALYSIS

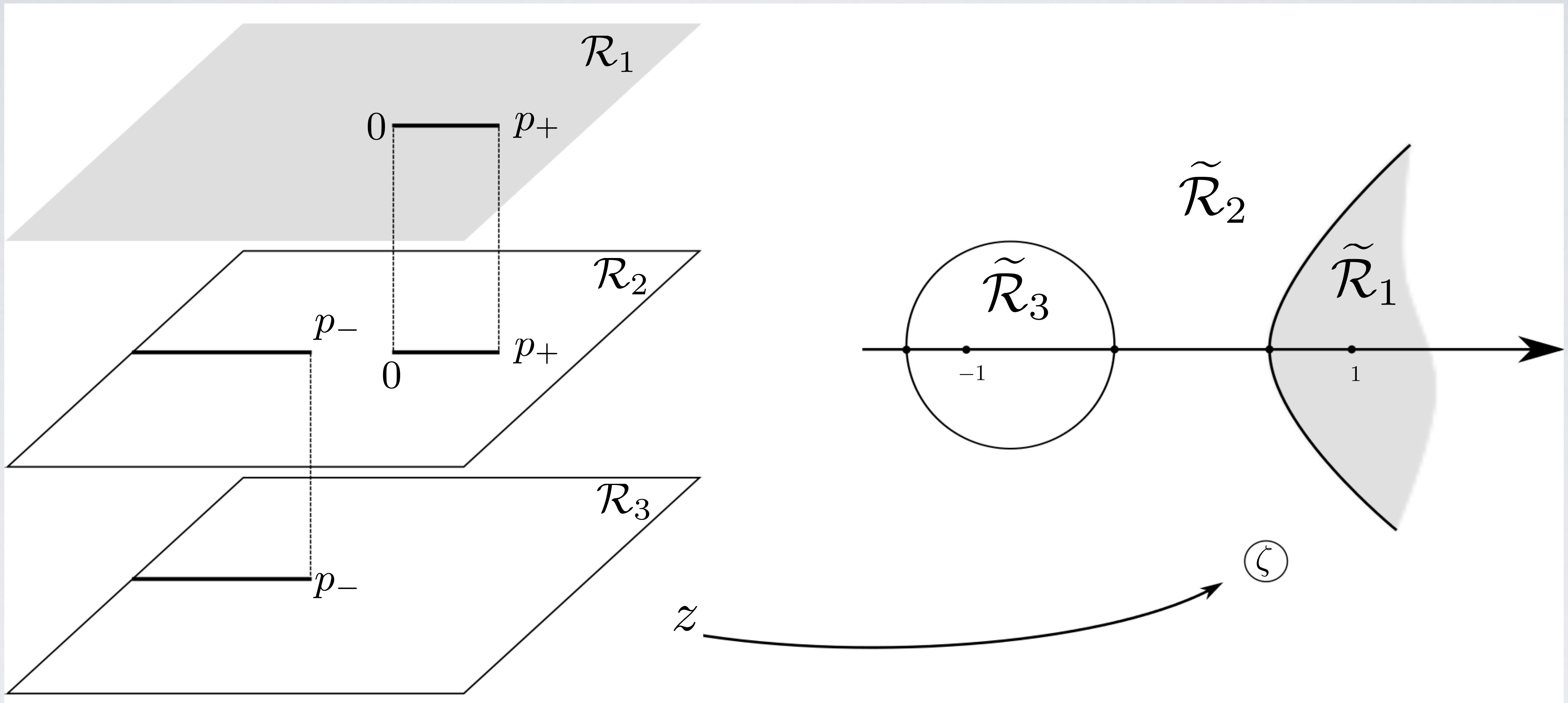


We seek $\widehat{\boldsymbol{N}}$ in the form

$$\widehat{\boldsymbol{N}}(z) = \begin{pmatrix} F_1(\zeta_1(z)) & F_1(\zeta_2(z)) & F_1(\zeta_3(z)) \\ F_2(\zeta_1(z)) & F_2(\zeta_2(z)) & F_2(\zeta_3(z)) \\ F_3(\zeta_1(z)) & F_3(\zeta_2(z)) & F_3(\zeta_3(z)) \end{pmatrix},$$

where F_j are functions in ζ with quadratic branching at the images of the branch points of $\mathcal{R}$, selected to satisfy the condition at infinity. They can be computed explicitly.

THE R-H STEEPEST DESCENT ANALYSIS



For example,

$$F_1(\zeta) = \frac{\zeta^{3/2}}{(\zeta^2+\zeta-1)^{1/2}}, \quad F_2(\zeta) = \frac{2^{-3/2} i\, \zeta^{1/2}(\zeta-1)}{(\zeta^2+\zeta-1)^{1/2}}, \ldots$$

They can be computed explicitly.

THE R-H STEEPEST DESCENT ANALYSIS

Step 5: local analysis at the end points 0 and $p_\pm$.
In order to get a close-to-$\boldsymbol{I}$ behavior of all jumps on the whole $\mathbb{C}$, we still need to solve a model problem at these points.
The solution is different from $\boldsymbol{N}$.

• At $p_\pm$ ("soft edges"), it is written in terms of the Airy function.

• At 0 ("hard edge"), it is written in terms of the modified Bessel functions I_0 and K_0, and the Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$.

This is more or less standard.

Step 6: last step!

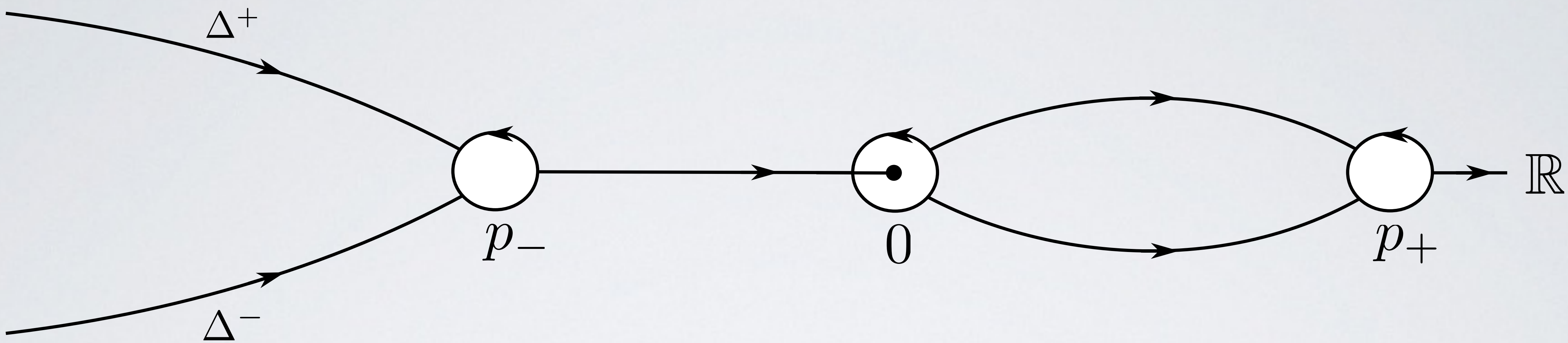
Using our "parametrices" (solutions to the model problems) we finally arrive at a RH problems for a matrix $\boldsymbol{R}$ whose jumps are $\mathcal{O}(1/n)$ close to $\boldsymbol{I}$ in $\mathbb{C}$ uniformly.

We conclude that $\boldsymbol{R}(z) = \boldsymbol{I} + \mathcal{O}\left(\frac{1}{n(|z|+1)}\right), \qquad n \to \infty$

If you are still with me, it is time to wrap up…

THE R-H STEEPEST DESCENT ANALYSIS

We have our plane $\mathbb{C}$ split into different regions by these contours:



In each domain, unravelling our transformations, we get an expression for the original matrix $\boldsymbol{Y}$.

For instance, assume that z lies in one of the unbounded component of the complement to these curves. Then

$$\boldsymbol{Y}(z) = \operatorname{diag}\left(e^{-n\omega}, 1, 1\right)\left(\boldsymbol{I} + \mathcal{O}\left(\frac{1}{n(|z|+1)}\right)\right)$$
$$\times \boldsymbol{N}(z)\operatorname{diag}\left(e^{n(g_1(z)+\omega)}, e^{-n(g_1(z)-g_2(z))}, e^{-ng_2(z)}\right)\left(\begin{array}{c:c} 1 & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & \boldsymbol{A}_{L,R}^{-1}(z) \end{array}\right)$$

THE R-H STEEPEST DESCENT ANALYSIS

It remains to see what this expression means for $\boldsymbol{Y}_{11} = P_{\vec{n}}$.

Here is a free sample:

Theorem: *Let*

$$\mathcal{H}(\zeta) = \frac{\zeta}{\sqrt{2}} \left(\frac{1+\zeta}{\zeta^2+\zeta-1} \right)^{1/2}, \quad \mathcal{H}(\infty) = 1$$

Then for $z \in \mathbb{C} \setminus [0, p_+]$, *with* $p_+ = \left(\frac{2}{\sqrt{5}-1}\right)^5$,

$$\boxed{Q_n(z) = e^{n g_1(z)} \mathcal{H}(\zeta_1(z)) \left(1 + \mathcal{O}\left(\frac{1}{n(|z|+1)} \right) \right)}$$

locally uniformly away from the interval $[0, p_+]$.

Recall that ζ_1 is the holomorphic branch of $\zeta(z)$ defined by

$$z = \frac{1+\zeta}{\zeta^2(1-\zeta)}, \quad \text{with } \zeta_1(\infty) = 1.$$

Also notice that $\left| e^{n g_1(z)} \right| = \exp\left(-V^{\lambda_1}(z)\right)$, as promised.

One more slide

THANK YOU

the audience and the organizers!