A Mahler-type inequality for Bergman kernels

partly joint work w/ B. Berndtsson and Y.A. Rubinstein

University of Maryland, College Park

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Aim of the talk

For a *convex body* $K \subset \mathbb{R}^n$ (that is a convex, compact set with non-empty interior) denote by

$$T_K := \mathbb{R}^n + \sqrt{-1}\,\mathrm{int}\,K \subset \mathbb{C}^n$$

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Theorem 1 (M.– Rubinstein, 2024)

For a symmetric convex body $K \subset \mathbb{R}^2$ (that is, $K = -K$),

$$\mathcal{B}_{T_K}(0,0) \geq \left(\frac{\pi}{4}\right)^2 \frac{1}{|K|^2}.$$

Bergman kernels

The Bergman kernel of a domain $\Omega \subset \mathbb{C}^n$

$$\mathcal{B}_\Omega : \Omega \times \Omega \to \mathbb{C}$$

is characterized by three properties:

- $\mathcal{B}_\Omega$ holomorphic in the first variable
- $\mathcal{B}_\Omega$ anti-holomorphic in the second variable
- $\mathcal{B}_\Omega$ enjoys the 'reproducing' property

$$\langle f, \mathcal{B}_\Omega(\cdot, w)\rangle_{L^2(\Omega)} = f(w), \quad \text{for all } L^2 \text{ holom. } f \text{ on } \Omega.$$

Notes:

- $\mathcal{B}_\Omega$ is real on the diagonal,

$$\mathcal{B}_\Omega(z,z) = \sup_{\substack{f \in L^2(\Omega) \\ \text{holomorphic}}} \frac{|f(z)|^2}{\|f\|_{L^2}^2}.$$

A conjecture of Błocki

This resolves the two-dimensional case of the decade-old conjecture:

Conjecture (Błocki 2014)

For a symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{B}_{T_K}(0,0) \geq \left(\frac{\pi}{4}\right)^n \frac{1}{|K|^2},$$

with equality obtained by the cube $[-1,1]^n$.

For reference:

$$\frac{\pi}{4} \approx 0.785.$$

Previous progress

Theorem 2 (Nazarov 2012)

For a symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{B}_{T_K}(0,0) \geq \left(\frac{\pi^2}{16}\right)^n \frac{1}{|K|^2}.$$

Note that

$$\frac{\pi^2}{16} \approx 0.617.$$

Idea of proof.

Use Hörmander's $\overline{\partial}$-theorem to construct a holomorphic function over T_K with good enough bounds on its L^2-norm. □

Previous progress

Theorem 3 (Berndtsson 2022)

For a symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{B}_{T_K}(0,0) \geq (0.623)^n \frac{1}{|K|^2}.$$

Idea of proof.

Twist the Bergman space to an 'easier' one. Get an easy estimate on the twisted space via the constant function. Use the plurisubharmonicty of Bergman kernels to obtain an estimate for the desired Bergman kernel. □

A (not so) small detour in convex geometry

What motivated Błocki's conjecture?

A (not so) small detour in convex geometry

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The answer to this question goes back to about 85 years ago. For any convex body we may define the polar (or dual) body via

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\},$$

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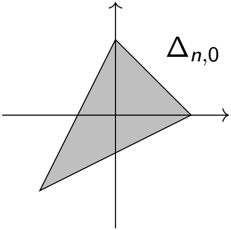

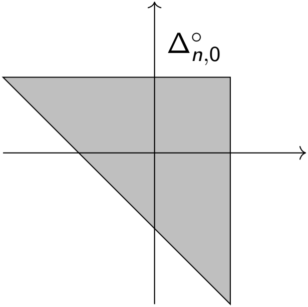

A (not so) small detour in convex geometry

Another way to define the polar body is as a sublevel set of the support function

$$h_K(y) := \sup_{x \in K} \langle x, y \rangle, \quad x \in K.$$

That is

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That is

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Utilizing this characterization, by a simple application of the fundamental theorem of calculus and Fubini's theorem, we obtain the following useful formula

$$\boxed{\int_{\mathbb{R}^n} e^{-h_K(y)} dy = n!|K^\circ|}$$

A (not so) small detour in convex geometry

In the 1930's, Kurt Mahler studied lattices and their duals aiming to extend Minkowski's famous theorem in the geometry of numbers concerning the existence of lattice points in convex bodies. As part of his work, Mahler needed to find a bound on the product of volumes:

$$|K||K^\circ|.$$

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For convenience, denote this product via

$$\mathcal{M}(K) := n!|K||K^\circ| = |K| \int_{\mathbb{R}^n} e^{-h_K(y)} dy,$$

called *the Mahler volume of* K (the constant $n!$ serves the purpose of tensoriality: $\mathcal{M}(K \times L) = \mathcal{M}(K)\mathcal{M}(L)$).

A (not so) small detour in convex geometry

- An important property of Mahler volume is its $GL(n,\mathbb{R})$ invariance. This comes from how polarity transforms under $GL(n,\mathbb{R})$ transformations:

$$(AK)^\circ = (A^{-1})^T K^\circ, \quad A \in GL(n,\mathbb{R}).$$

Therefore,

$$\begin{aligned}
\mathcal{M}(AK) &= n!|AK||(A^{-1})^T K^\circ| \\
&= n!|\det A||K||\det A|^{-1}|K^\circ| \\
&= n!|K||K^\circ| \\
&= \mathcal{M}(K).
\end{aligned}$$

In contrast, $\mathcal{M}$ is **not** invariant under translations.

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In contrast, $\mathcal{M}$ is **not** invariant under translations.

- Another important property is that polarity is an involution for convex bodies that contain the origin in their interior:

$$(K^\circ)^\circ = K.$$

A (not so) small detour in convex geometry

Using a precursor to John's theorem, Mahler obtained crude bounds

$$\mathcal{M}(K) \geq \frac{4^n}{n!}$$

and conjectured the following:

Conjecture (Mahler, 1938-39)

For a symmetric convex body $K \subset \mathbb{R}^n$,

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On the same year (1939), but in a different paper, Mahler proved his conjecture in dimension $n = 2$. The three-dimensional case was resolved by Iriyeh–Shibata 80 years later! The conjecture remains open for $n \geq 4$.

Mahler's 2D proof

The proof is via sliding of a vertex: Let S be a symmetric convex polytope.

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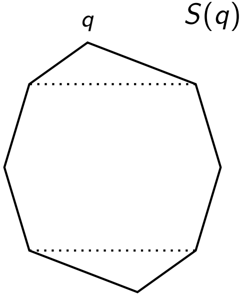
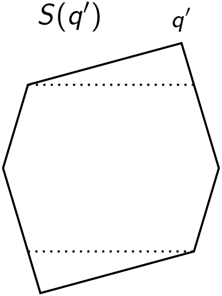
- Fix a vertex q.
- Slide the vertex q and its antipodal $-q$ in a manner parallel to their adjacent vertices, so that the total volume remains the same.



(a) Before sliding



(b) After sliding

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- Note that exactly 4 vertices move in the polar $S(q)^\circ$. By a direct calculation

$$q \mapsto \frac{1}{|S(q)^\circ|}$$

  can be shown to be a convex quadratic polynomial of the sliding parameter.

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- Note there are exactly two distinct positions q' and q'' for which $S(q')$ and $S(q'')$ have exactly two vertices less than the initial position.
- By convexity,
$$|S(q)^\circ| \geq \min\{|S(q')|, |S(q'')|\},$$
hence the claim.

Progress towards Mahler

Mahler's conjecture has motivated a lot of work. Let us only mention two important results:

Theorem 4 (Bourgain–Milman, 1987)

There exists a constant $c > 0$ *such that for all* $n \in \mathbb{N}$ *and convex bodies* $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq c^n$$

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The significance of this theorem is it obtains the correct behavior 'asymptotically'.
Best bound so far:

Theorem 5 (Kuperberg 2008, Berndtsson 2021)

For a symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq \pi^n$$

.

Nazarov's approach

A very interesting approach to the Bourgain–Milman inequality is due to Nazarov who observed the following inequality between Mahler volume and Bergman kernels of tube domains:

$$\mathcal{M}(K) \geq \pi^n |K|^2 \mathcal{B}_{T_K}(0,0)$$

for convex bodies with their barycenter at the origin.

Corollary 6 (Nazarov 2012)

For a symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq \left(\frac{\pi^3}{16}\right)^n .$$

Note $\pi^3/16 \approx 1.937$.

Nazarov's proof

To obtain the lower bound on the Mahler volume via the Bergman kernel, Nazarov capitalizes on an explicit formula for the Bergman kernel of tube domains due to Rothaus (1960), Korányi (1962), and Hsin (2005):

$$\mathcal{B}_{T_K}(0,0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{dy}{J_K(y)}, \quad \text{where } J_K(y) := \int_K e^{-2\langle x,y\rangle} dx. \tag{1}$$

More generally,

$$\mathcal{B}_{T_K}(z,w) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{\sqrt{-1}\langle z-\overline{w},y\rangle}}{J_K(y)} dy$$

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Claim $J_K(y) \geq |K| e^{h_K(-y)}/2^n$ when $b(K) := \int_K x \frac{dx}{|K|} = 0$.

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Claim $J_K(y) \geq |K| e^{h_K(-y)}/2^n$ when $b(K) := \int_K x \frac{dx}{|K|} = 0$. *Proof.* Jensen's inequality + convexity.
(1) + Claim $\Longrightarrow$
$\mathcal{M}(K) = |K| \int_{\mathbb{R}^n} e^{-h_K(-y)} dy \geq \frac{|K|^2}{2^n} \int_{\mathbb{R}^n} \frac{dy}{J_K(y)} = \pi^n |K|^2 \mathcal{B}_{T_K}(0,0).$

Our idea (2022, 2023)

Rewrite the Rothaus–Korányi–Hsin formula

$$\mathcal{B}_{T_K}(z,w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle z-\overline{w},y\rangle - \log J_K(y)} dy.$$

Since $\mathcal{M}$ is related to $\mathcal{B}_{T_K}(0,0)$ by relating J_K to h_K, perhaps J_K is some sort of support function itself?

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Definition (M.–Rubinstein, 2022)

The L^1-support function

$$h_{1,K}(y) := \log \int_K e^{\langle x,y\rangle}\frac{dx}{|K|}.$$

Our idea (2022, 2023)

Now, since $\log J_K(y) = h_{1,K}(2y)$,

$$\mathcal{B}_{T_K}(z,w) = \frac{(2\pi)^{-n}}{|K|} \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle z-\overline{w},y\rangle - h_{1,K}(2y)} dy,$$

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or, by a change of variables,

$$\mathcal{B}_{T_K}(z,w) = \frac{(4\pi)^{-n}}{|K|} \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle \frac{z-\overline{w}}{2}, y\rangle - h_{1,K}(y)} dy.$$

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When $z = w = 0$ (or, more generally, $z = w$) this becomes

$$\mathcal{B}_{T_K}(0,0) = \frac{(4\pi)^{-n}}{|K|} \int_{\mathbb{R}^n} e^{-h_{1,K}(y)} dy.$$

Our idea (2022, 2023)

Definition (Berndtsson–M.–Rubinstein, 2023)

The L^1-Mahler volume

$$\mathcal{M}_1(K) := |K| \int_{\mathbb{R}^n} e^{-h_{1,K}(y)} dy$$

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The L^1-Mahler volume

$$\mathcal{M}_1(K) := |K| \int_{\mathbb{R}^n} e^{-h_{1,K}(y)} dy$$

Consequently,

$$|K|^2 \mathcal{B}_{T_K}(0,0) = (4\pi)^{-n} \mathcal{M}_1(K).$$

So, Nazarov's inequality, $\mathcal{M}(K) \geq \pi^n |K|^2 \mathcal{B}_{T_K}(0,0)$ becomes

$$\mathcal{M}(K) \geq \frac{\mathcal{M}_1(K)}{4^n}.$$

Back to the problem

With our definitions, Błocki's conjecture then becomes L^1-version of the Mahler conjecture:

Conjecture

For a symmetric convex body $K \subset \mathbb{R}^n$, $\mathcal{M}_1(K) \geq \mathcal{M}_1([-1,1]^n)$.

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Question: Can we apply Mahler's sliding argument to prove Błocki's conjecture?

Not yet. Recall that Mahler's proof involved a careful study of the polar body under sliding.

Geometric interpretation

Does $\int_{\mathbb{R}^n} e^{-h_{1,K}}$ have a geometric meaning? To rephrase the question, we know $\int_{\mathbb{R}^n} e^{-h_K(y)} dy = n!|K^\circ|$, is there a similar interpretation of our integral? Is it the volume of a canonical convex body?

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Theorem-Definition (B.–M.–R., 2023)

$$\mathcal{M}_1(K) = n!|K||K^{\circ,1}|$$

where $K^{\circ,1}$ is convex, closed, with non-empty interior, and

$$K^{\circ,1} = \{y \in \mathbb{R}^n : \|y\|_{K^{\circ,1} \leq 1}\},$$

where

$$\|y\|_{K^{\circ,1}} := \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_{1,K}(ry)}dr\right)^{-\frac{1}{n}}$$

is a near-norm (norm if $K = -K$). Moreover, $K^{\circ,1}$ is compact if and only if $0 \in \operatorname{int} K$, and if K is symmetric then so is $K^{\circ,1}$.

Convex geometric interpretation of Bergman kernel

Corollary

$$\mathcal{B}_{T_K}(0,0) = \frac{n!}{(4\pi)^n} \frac{|K^{\circ,1}|}{|K|}.$$

L^p-polarity

More generally, we can join $\mathcal{M}$ and $\mathcal{M}_1$, h_K and $h_{1,K}$ via a 1-parameter family of Mahler volumes, support functions:

Definition (B.–M.–R., 2023)

For $p \in (0, \infty)$,

- $$h_{p,K}(y) := \log \left[\int_K e^{p\langle x,y\rangle} \frac{dx}{|K|} \right]^{\frac{1}{p}}, \quad y \in \mathbb{R}^n$$
- $$\mathcal{M}_p(K) := |K| \int_{\mathbb{R}^n} e^{-h_{p,K}(y)} dy \overset{Thm}{=} n!|K||K^{\circ,p}|.$$

Note $K \mapsto K^{\circ,p}$ is called "L^p-polarity", however it is not a duality operation.

L^p-Mahler conjecture

Conjecture (B.–M.–R., 2023)

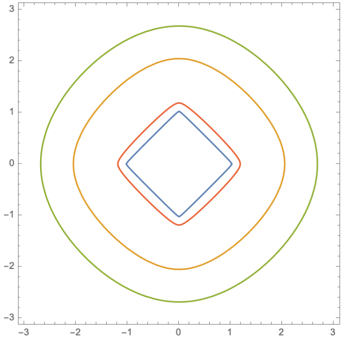
For $p \in (0, \infty)$ and symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}_p(K) \geq \mathcal{M}_p([-1, 1]^n),$$

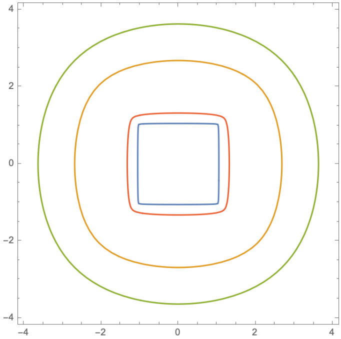
with equality if and only if $K = A[-1, 1]^n$ for some $A \in GL(n, \mathbb{R})$.

L^p-polarity

Proof by picture that L^p-polarity is not duality: the L^p-polar body is always smooth, so starting from a cube, the L^p-double polar cannot be the cube...



(a) $(B^2_\infty)^{\circ,p}$



(b) $(B^2_1)^{\circ,p}$

Proof of the two-dimensional L^p-Mahler conjectures

Błocki's conjecture is obtained as the special $p = 1$ case in the following:

Theorem 7 (M.–R., 2024)

Let $p \in (0, \infty)$. For a symmetric convex body $K \subset \mathbb{R}^2$,

$$\mathcal{M}_p(K) \geq \mathcal{M}_p([-1, 1]^2).$$

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Proof.

- Start with a symmetric convex polytope $S \subset \mathbb{R}^2$ and fix a vertex q to perform the sliding.

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Proof.

- Start with a symmetric convex polytope $S \subset \mathbb{R}^2$ and fix a vertex q to perform the sliding.
- As in the $p = \infty$ case, the claim can be reduced to proving that

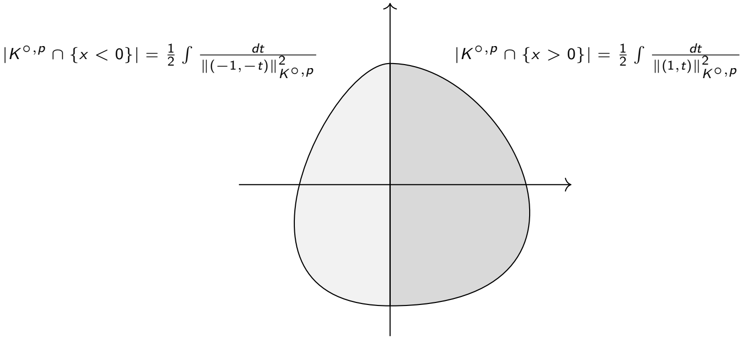
$$q \mapsto \frac{1}{|S(q)^{\circ,p}|}$$

is a convex function of q.

Proof of the two-dimensional L^p-Mahler conjectures

- We need a more convenient formula for the volume of the L^p-polar body. By splitting the L^p-polar into two hemispheres and then using polar coordinates:

$$|K^{\circ,p}| = \frac{1}{2}\int_{\mathbb{R}} \frac{dt}{\|(1,t)\|_{K^{\circ,p}}^2} + \frac{1}{2}\int_{\mathbb{R}} \frac{dt}{\|(-1,-t)\|_{K^{\circ,p}}^2}.$$



Proof of the two-dimensional L^p-Mahler conjectures

- In particular, since $S(q)$ is symmetric

$$|S(q)^{\circ,p}| = \int_{\mathbb{R}} \frac{dt}{\|(1,t)\|^2_{S(q)^{\circ,p}}}.$$

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$$(t,q) \mapsto \|(1,t)\|_{S(q)^{\circ,p}}.$$
- By a theorem of Ball we reduce this to showing a type of "strong joint convexity" for the L^p-support functions of the 1-parameter family of sliding polytopes
$$h_{p,S\left(\frac{q+q'}{2}\right)}\left(r\left(1,\frac{t+t'}{2}\right)\right) \le \frac{s}{\tau+s} h_{p,S(q)}\left(\tau(1,t)\right) + \frac{\tau}{\tau+s} h_{p,S(q')}\left(s(1,t')\right)$$
for all $r, \tau, s > 0$ such that $\frac{1}{r} = \frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{s}\right)$. □

Implications for Mahler's conjecture

A resolution of Błocki's conjecture would not necessarily imply Mahler. Indeed, Nazarov's inequality would give

$$\mathcal{M}(K) \geq \left(\frac{\pi^2}{4}\right)^n$$

which is 'only' $\pi^2/4 \approx 2.467$. This is because Nazarov's approach is an approach to the L^1-Mahler conjecture.

Complex-convex approach to Mahler conjecture

The gap between L^1 and L^∞ should also be bridged:

Conjecture (M.–R.)

For a symmetric convex body,

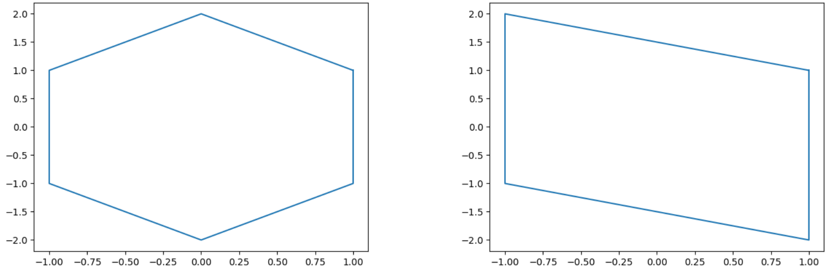
$$\frac{\mathcal{M}(K)}{\mathcal{M}_1(K)} \geq \frac{\mathcal{M}([-1,1]^n)}{\mathcal{M}_1([-1,1]^n)} = \left(\frac{4}{\pi^2}\right)^n.$$

$\mathcal{M}/\mathcal{M}_1$ under sliding

Consider the family of hexagons

$$H(t) := \operatorname{co}\{(1,1),(t,2),(-1,1),(-1,-1),(-t,-2),(1,-1)\}$$

that arise by the sliding of two antipodal vertices of $H(0)$ so that the volume is kept constant.



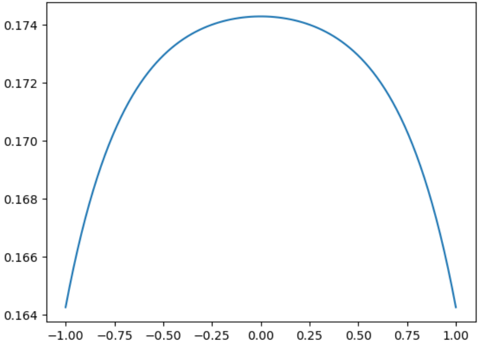
(a) $t = -1$

(b) $t = 0$

Figure: Hexagon under sliding of antipodal vertices

$\mathcal{M}/\mathcal{M}_1$ under sliding

The following graph depicts $\mathcal{M}/\mathcal{M}_1$ for a hexagon under sliding of two antipodal vertices, reducing it to a square.



Non-symmetric analogues

The definitions of the L^p-support functions and L^p-polarity do not require any symmetry and make sense for all convex bodies.
For example, for the simplices

$$\Delta_{n,0} := \mathrm{conv}\{-\sum_{i=1}^{n} e_i, e_1, \ldots, e_n\} \quad \text{and} \quad \Delta_{n,+} := \mathrm{conv}\{0, e_1, \ldots, e_n\}$$

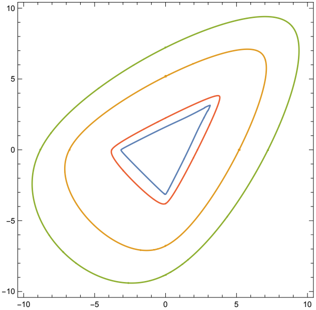


Figure: $(\Delta_{n,0})^{\circ,p}$

Non-symmetric analogues

Nazarov's inequality bounding the Mahler volume from below by the Bergman kernel of the tube domain evaluated at the origin holds for all convex bodies with barycenter at the origin. In addition,

Theorem 8 (M.–R., 2022)

For a convex body $K \subset \mathbb{R}^n$,

$$|K|^2 \mathcal{B}_{T_K}(0,0) \geq 4^{-n}.$$

Corollary 9 (M.–R., 2022)

For a convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq \left(\frac{\pi}{4}\right)^n.$$

Non-symmetric analogues

However, this is sub-optimal to what one gets from the Kuperberg–Berndtsson bound with symmetrization:

Theorem 10 (Kuperberg 2008, Berndtsson 2021)

For a convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq \left(\frac{\pi}{2}\right)^n.$$

This is the best bound so far regarding the non-symmetric version of Mahler's conjecture:

Conjecture (Mahler)

For a convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}(K) \geq \mathcal{M}(\Delta_{n,0}) = \frac{(n+1)^{n+1}}{n!} = e^{n+o(n)}.$$

Non-symmetric analogues

In a similar manner:

Conjecture (B.–M.–R. 2023)

Let $p \in (0, \infty)$. For a convex body $K \subset \mathbb{R}^n$,

$$\mathcal{M}_p(K) \geq \inf_{x \in \mathbb{R}^n} \mathcal{M}_p(\Delta_{n,0} - x) = \mathcal{M}_p(\Delta_{n,0}).$$

Proof in dimension two

Theorem 11 (M.–R. 2024)

Let $p \in (0, \infty)$. For $K \subset \mathbb{R}^2$,

$$\mathcal{M}_p(K) \geq \mathcal{M}_p(\Delta_{2,0}).$$

Proof.

- Start with a convex polytope $P \subset \mathbb{R}^2$ and fix a vertex q to perform the sliding.

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Proof.

- Start with a convex polytope $P \subset \mathbb{R}^2$ and fix a vertex q to perform the sliding.
- As in the symmetric case, the claim can be reduced to proving that

$$q \mapsto \frac{1}{|P(q)^{\circ,p}|}$$

  is a convex function of q.

Proof in dimension two

- Recall
$$|P(q)^{\circ,p}| = I_+(q) + I_-(q)$$
where
$$I_+(q) = |P(q)^{\circ,p} \cap \{x > 0\}| = \frac{1}{2}\int_{\mathbb{R}} \frac{dt}{\|(1,t)\|^2_{P(q)^{\circ,p}}}$$
and
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- By the previous (using the Borell–Brascamp–Lieb and Ball inequalities),
$$q \mapsto \frac{1}{I_+(q)} \quad \text{and} \quad q \mapsto \frac{1}{I_-(q)}$$
are convex.

Proof in dimension two

- **Problem** This does **not** mean that

$$q \mapsto \frac{1}{|P(q)^{\circ,p}|} = \frac{1}{I_+(q) + I_-(q)}$$

is convex. Except! if, in addition, the ratio

$$q \mapsto \frac{I_+(q)}{I_-(q)}$$

remains constant throughout this motion.

Proof in dimension two

- To achieve a constant ratio throughout the sliding, we introduce a translating motion, independent of the sliding. Consider

$$(q, x) \mapsto P(q) - (x, 0).$$

Essentially by the Intermediate value theorem, there is

$$x = x(p)$$

that keeps the ratio

$$\frac{|(P(q) - (x(p), 0))^{\circ, p} \cap \{x > 0\}|}{|(P(q) - (x(p), 0))^{\circ, p} \cap \{x < 0\}|}$$

constant throughout the sliding. □

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