Riesz Energy with an External Field: Dimensionality of Minimizers

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Energy

Given lower semi-continuous functions $W : \mathbb{R}^d \to (-\infty, \infty]$ and $V : \mathbb{R}^d \to (-\infty, \infty]$, the **(continuous) energy** of a Borel probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is

$$I_{W,V}(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(W(x-y) + V(x) + V(y) \Big) d\mu(x) d\mu(y).$$

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- Sufficient growth (attraction) conditions on W or V as $\|x\| \to \infty$, or restriction of the problem to a compact space A (i.e. $V(x) = \infty$ for $x \notin A$), usually guarantees the existence of (compactly supported) minimizers.

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- What can we say about the dimension of the support of a minimizer? General trend: Stronger (local) repulsion $\Rightarrow$ higher dimension, stronger attraction $\Rightarrow$ lower dimension

Modeling Natural Systems

The evolution of a system can be described by the aggregation equation

$$\frac{\partial \mu_t(x)}{\partial t} = \nabla \cdot \mu_t(x) \nabla \left(\left(\int_{\mathbb{R}^d} W(x-y)\, d\mu_t(y) \right) + V(x) \right)$$

with interaction kernel W and external field V. Then (local) minimizers of $I_{W,V}$ are steady states of the gradient flow.

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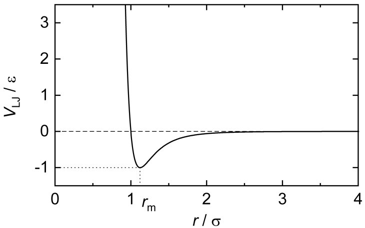


For swarm behavior, neutral particles, or cell interactions, one may use repulsive-attractive kernels, like **Lennard-Jones/power law potentials**

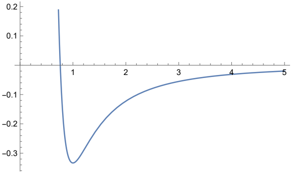
$$W_{s,\alpha}(x-y) = \frac{\|x-y\|^{-s}}{s} + \frac{\|x-y\|^{\alpha}}{\alpha}.$$

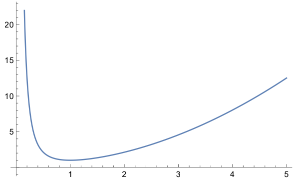
Existence and Compactness of Minimizers

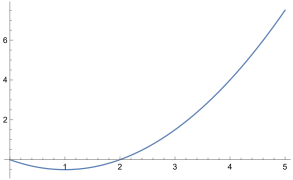
Lennard-Jones/power law potentials

$$W_{s,\alpha}(x-y) = \frac{\|x-y\|^{-s}}{s} + \frac{\|x-y\|^{\alpha}}{\alpha}, \quad -\infty < s < d, \; -s < \alpha < \infty,$$

with conventions $\frac{\|x-y\|^{-0}}{0} = -\log(\|x-y\|)$.







Theorem (Cañizo, Carrillo, Patacchini '15)

If $s < d$ and $\alpha > -s$, then there exists a minimizer of $I_{W_{s,\alpha}}$. Moreover, there exists some $K \in (0,\infty)$, such that if μ_{eq} is a minimizer, then $\operatorname{diam}(\operatorname{supp}(\mu_{eq})) \le K$.

Repulsion Affecting Dimension

Lennard-Jones/power law potentials

$$W_{s,\alpha}(x-y) = \frac{\|x-y\|^{-s}}{s} + \frac{\|x-y\|^{\alpha}}{\alpha}, \quad s \in (-\infty, d), \alpha \in (-s, \infty].$$

Theorem (Carrillo, Figalli, Patachini '17)

If $s < -2$, $-s < \alpha < \infty$, and μ_{eq} is a minimizer of $I_{W_{s,\alpha},0}$, then $\mathrm{supp}(\mu_{eq})$ is finite.

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Theorem (Balague, Carrillo, Laurent, Raoul '13)

If $-2 < s < d-2$, $-s < \alpha < \infty$, and μ_{eq} is a minimizer of $I_{W_{s,\alpha},0}$, then $\dim(\text{supp}(\mu_{eq})) \geq s+2$.

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Theorem (Carrillo, Delgadino, Mellet '16)

If $d-2 \leq s < d$, $0 < \alpha < \infty$, and μ_{eq} is a minimizer of $I_{W_{s,\alpha},0}$, then μ is absolutely continuous with respect to the Lebesgue measure, i.e. $\dim(\operatorname{supp}(\mu_{eq})) = d$.

Attraction Affecting Dimension

Theorem (Davies, Lim, McCann, ’22, ’23)

For $d \geq 2$ and $s = -2$,

- *if $2 < \alpha < 4$, then μ_{eq} is the uniform measure on some sphere, i.e.* $\dim(\mathrm{supp}(\mu_{eq})) = d - 1$.
- *If $4 < \alpha$, μ_{eq} is the uniform measure on the vertices of a regular simplex, i.e.* $\dim(\mathrm{supp}(\mu_{eq})) = 0$.
- *if $\alpha = 4$,* $\dim(\mathrm{supp}(\mu_{eq})) \in [0, d - 1]$.

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Theorem (Carrillo, Shu, '23; Frank, M. '24)

For $d \geq 2$ and $d - 4 < s < \frac{d^2-3d-2}{d+1} < d - 3$

- *if $\alpha = 2$, then μ_{eq} is absolutely continuous and supported on a ball, i.e.* $\dim(\text{supp}(\mu_{eq})) = d$.
- *if $\alpha = 4$, then μ_{eq} is the uniform measure on some sphere, i.e.* $\dim(\text{supp}(\mu_{eq})) = d - 1$

Riesz s-energies

Riesz s-kernels

For $s \in \mathbb{R}$, we define the **Riesz s-kernel** as

$$W_s(x-y) = \begin{cases} \frac{1}{s}\|x-y\|^{-s} & s \neq 0 \\ -\log(\|x-y\|) & s = 0 \end{cases}.$$

Generally interested in minimizing

$$I_{W_s,V}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{1}{s}\|x-y\|^{-s} + V(x) + V(y)\right) d\mu(x)d\mu(y).$$

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Particularly interested in minimizing, for $-2 < s < d$, $\max\{0, -s\} \leq \alpha < \infty$, and $\gamma > 0$

$$I_{W_s,V_{\alpha,\gamma}}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{1}{s}\|x-y\|^{-s} + \gamma\frac{\|x\|^\alpha}{\alpha} + \gamma\frac{\|y\|^\alpha}{\alpha}\right) d\mu(x)d\mu(y).$$

Some Connections

Theorem (McCarthy '23)

Let $X_n = (X^{(1,n)}, X^{(2,n)}, ..., X^{(d,n)})$ *be a d-tuple of* $n \times n$ *commuting, Hermitian complex random matrices. Then, as* $n \to \infty$, *the distribution of eigenvalues of* $\frac{1}{\sqrt{n}} X_n$ *will tend to the minimizing measure of* $I_{W_0, V_{2,2}}$.

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Theorem (Hertrich, Gräf, Beinert, Steidl '24; CMSVW)

The minimizer of $I_{W_s, V_{\alpha,\gamma}}$ is also the minimizer of

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_s(x-y)\, d\mu(x)\, d\mu(y) + \frac{\gamma}{\alpha} D_\alpha^\alpha(\delta_0, \mu)$$

where D_α is the Wasserstein α-metric.

The minimizer is how charge distributes from being concentrated at the origin under a Wasserstein steepest descent flow after a certain amount of time.

Our Main Potential Theoretic Tools

Conditional Strict Positive Definiteness

For compact $A \subset \mathbb{R}^d$, we call a symmetric, lower semi-continuous kernel $K : A \times A \to (-\infty, \infty]$ **conditionally strictly positive definite** if for all finite signed Borel measures μ such that $\mu(A) = 0$ and $\mu \not\equiv 0$,

$$I_K(\mu) := \int_A \int_A K(x,y) d\mu(x) d\mu(y) > 0.$$

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Lemma

For $-2 < s < d$, the kernel

$$K_{s,V}(x,y) = W_s(x-y) + V(x) + V(y)$$

is conditionally strictly positive definite on every compact $A \subset \mathbb{R}^d$ (we call this being conditionally strictly positive definite on $\mathbb{R}^d$).

Our Main Potential Theoretic Tools

Theorem

Suppose K is conditionally strictly positive definite on $\mathbb{R}^d$ and a minimizer of I_K exists. Then μ_{eq} minimizes I_K if and only if there is some constant C, such that

$$U_K^{\mu_{eq}}(x) := \int_{\mathbb{R}^d} K(x,y)\, d\mu_{eq}(y) \begin{cases} = C & x \in \operatorname{supp}(\mu) \\ \geq C & x \in \mathbb{R}^d \end{cases}. \tag{1}$$

Moreover, the μ_{eq} uniquely minimizes I_K.

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Corollary

If $-2 < s < d$, and a minimizer μ_{eq} of $I_{W_s,V}$ exists, then it is unique and the only measure in $\mathcal{P}(\mathbb{R}^d)$ to satisfy (1).

Simplifying with Radial Symmetry

If $-2 < s < d$, V is radially symmetric, i.e. $V(x) = f(\|x\|^2)$, then μ_{eq} is radially symmetric, as for any $g \in \mathrm{O}(d)$,

$$I_{W_s,V}(g_{\#}\mu_{eq}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\frac{1}{s}\|gx-gy\|^{-s} + f(\|gx\|^2) + f(\|gy\|^2)\Big)d\mu_{eq}(x)d\mu_{eq}(y)$$
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Theorem (Saff, Totik '97; Boradochov, Hardin, Saff '19; Dragnev, Orive, Saff, Wielonsky '23)

Let $f(\infty) := \lim_{r\to\infty} f(r^2)$. There exists a unique compactly supported minimizer of $I_{W_s,V}$ if

- $0 < s < d$ *and* $\lim_{r\to\infty} sr^s\left(f(r^2) - f(\infty)\right) < -1$,
- $s = 0$ *and* $\lim_{r\to\infty} \left(f(r^2) - \log(r)\right) = \infty$,
- $-2 < s < 0$ *and* $\limsup_{r\to\infty} sr^s f(r^2) < -2^{-s}$.

A One-dimensional Problem

When V and μ_{eq} are radial, we can separate the radial and angular parts of the measure, so

$$g_{s,V}(\|x\|) := U^{\mu_{eq}}_{W_s,V}(x) = f(\|x\|^2) + \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{s}\|x - Ry\|^{-s} d\sigma(y) d\nu(R).$$

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For $R > 0$,

$$\int_{\mathbb{S}^{d-1}} \frac{1}{s}\|x - Ry\|^{-s} d\sigma(y) = \begin{cases} \frac{1}{s} R^{-s} \, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix} \middle| \frac{\|x\|^2}{R^2} \right), & \|x\| \leq R \\ \frac{1}{s} \|x\|^{-s} \, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix} \middle| \frac{R^2}{\|x\|^2} \right) & \|x\| \geq R \end{cases}.$$

(Get a slightly different function for the log case)

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Frostman conditions:

$$g_{s,V}(\tau) + \int_{\mathbb{R}^d} V(y) d\mu_{eq}(y) \begin{cases} = I_{s,V}(\mu_{eq}) & \tau \in \operatorname{supp}(\nu) \\ \geq I_{s,V}(\mu_{eq}) & \tau \in [0, \infty) \end{cases}.$$

Sphere as Minimizer

For $-2 < s < d-1$ and some $R = R_{s,V} > 0$, and setting $t = \frac{\|x\|^2}{R^2}$, we find that σ_R satisfying the Frostman conditions is equivalent to

$$h_{s,V}(t) := U^{\sigma_R}_{W_s,V}(x) = \begin{cases} \frac{R^{-s}}{s}\, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix} \middle| \, t \right) + f(R^2 t) & t \in [0,1] \\ \frac{R^{-s}}{s} t^{-s/2}\, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix} \middle| \, \frac{1}{t} \right) + f(R^2 t) & t \in [1,\infty) \end{cases}$$

having its minimum at 1.

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The radius $R_{s,V}$ is a solution to

$$R^{s+2} f'(R^2) = {}_2\mathrm{F}_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix} \middle| 1\right).$$

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Theorem (Chafaï, M., Saff, Vu, Womersley)

If $-2 < s \leq d-4$ and $f''(t) \geq 0$ for all $t \in [0,\infty)$, $\mu_{eq} = \sigma_{R_{s,V}}$.

Sphere as a Minimizer

Theorem (Chafaï, M., Saff, Vu, Womersley)

Let $-2 < s < d-3$, $\gamma > 0$ *and*

$$\alpha \geq \alpha_{s,d} := \max\left\{ \frac{s\,{}_2F_1\left(\begin{matrix}\frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2}\end{matrix} \middle| 1\right)}{2 - 2\,{}_2F_1\left(\begin{matrix}\frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2}\end{matrix} \middle| 1\right)}, 2 - \frac{(s+2)(d-s-4)}{2(d-s-3)} \right\}.$$

Then, with $R = \left(\frac{\Gamma(\frac{d}{2})\Gamma(d-s-1)}{2\gamma\Gamma(\frac{d-s}{2})\Gamma(d-1-\frac{s}{2})}\right)^{\frac{1}{\alpha+s}}$, σ_R *uniquely minimizes*

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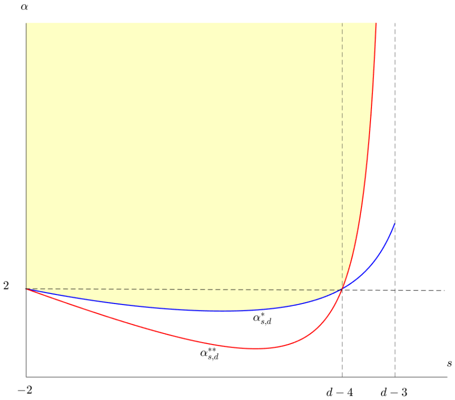
$$\alpha \geq \alpha_{s,d} := \max\left\{ \frac{s\, {}_2F_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix}\middle| 1\right)}{2 - 2\, {}_2F_1\left(\begin{matrix} \frac{s}{2}, \frac{2+s-d}{2} \\ \frac{d}{2} \end{matrix}\middle| 1\right)}, 2 - \frac{(s+2)(d-s-4)}{2(d-s-3)} \right\}.$$

Then, with $R = \left(\frac{\Gamma(\frac{d}{2})\Gamma(d-s-1)}{2\gamma\Gamma(\frac{d-s}{2})\Gamma(d-1-\frac{s}{2})}\right)^{\frac{1}{\alpha+s}}$, σ_R *uniquely minimizes*

$$I_{W_s, V_{\alpha,\gamma}}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{1}{s}\|x-y\|^{-s} + \gamma\frac{\|x\|^\alpha}{\alpha} + \gamma\frac{\|y\|^\alpha}{\alpha}\right) d\mu_{eq}(x)\, d\mu_{eq}(y).$$

The bound on α is sharp. We expect for $-s < \alpha < \alpha_{s,d}$, $\dim(\mathrm{supp}(\mu_{eq})) = d$.

Sphere as a Minimizer



$$\alpha_{s,d} = \max\{\alpha^*_{s,d}, \alpha^{**}_{s,d}\}$$

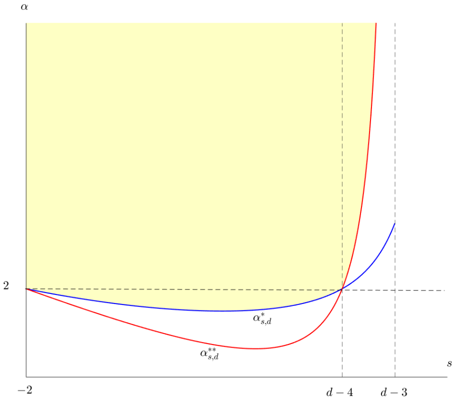
with

$$\alpha^*_{s,d} := \frac{s \cdot {}_2F_1\left(\frac{s}{2}, \frac{2+s-d}{2}; \frac{d}{2}; 1\right)}{2 - 2 \cdot {}_2F_1\left(\frac{s}{2}, \frac{2+s-d}{2}; \frac{d}{2}; 1\right)}$$

and

$$\alpha^{**}_{s,d} := 2 - \frac{(s+2)(d-s-4)}{2(d-s-3)},$$

Sphere as a Minimizer



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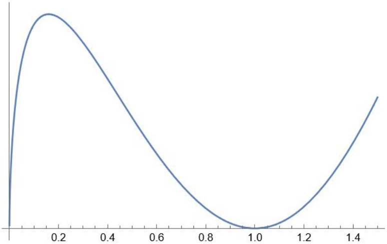
$$\alpha^{**}_{s,d} := 2 - \frac{(s+2)(d-s-4)}{2(d-s-3)},$$

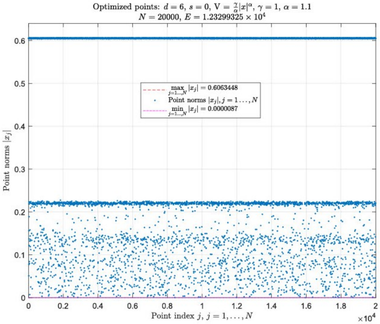
σ_R is never a minimizer for $s \geq d-3$. If $d-4 < s < d-3$ and $\alpha < \alpha_{s,d}$, $h_{s,V_{\alpha,\gamma}}(t)$ is not convex at 1, so σ_R is not a minimizer, and we expect a minimizer supported on an annulus.

Mixed Support

For $-2 < s \leq d - 4$ and $\alpha < \alpha_{s,d}$, $U^{\sigma_R}_{W_s,V}$ achieves its minimum at 0, so σ_R is not μ_{eq}. The minimizer seems more likely to be a combination of an absolutely continuous measure on a ball and singular measures on spheres on or outside the boundary.



$h_{s,V}(t)$ for $\alpha = \alpha^*_{s,d}$

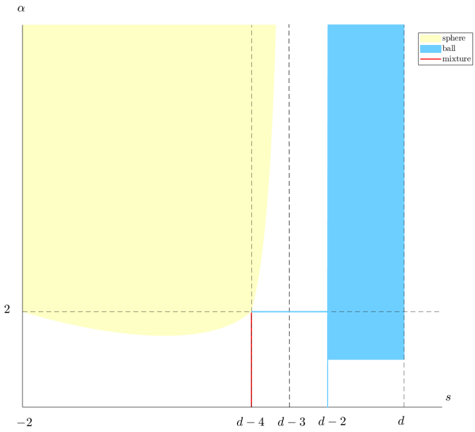


Numerics for $\alpha = 1.1$, $d = 6$, $s = d - 6 = 0$.

Known Minimizers

Theorem (Chafaï, M., Saff, Vu, Womersley)

If $d-3 \leq s < d$, and f is $\mathcal{C}^2$ (in the extended sense) on $[0,\infty)$, with f'' finite on $(0,\infty)$, and such that μ_{eq} exists. Then $\operatorname{supp}(\mu_{eq})$ is the union of uncountably many spheres.



Mhaskar, Saff ’92; Caffarelli, Vázquez ’01; López-García ’10; Bilogliadov ’18; Chafaï, Saff, Womersley ’22,’23; Carrillo, Shu ’23; Hertrich, Gräf, Beinert, Steidl ’24; Chafaï, M., Saff, Vu, Womersley.

Density of Measure from External Field

For $0 < s < d$, the Riesz Potential operator

$$(-\Delta)^{-\frac{d-s}{2}} V(x) := C_{d,s} \int_{\mathbb{R}^d} \frac{V(y)}{\|x-y\|^s} dy$$

acts as the inverse of the fractional Laplacian $(-\Delta)^{\frac{d-s}{2}}$.

$$\Delta V(x) = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} V(x), \quad (-\Delta)^{\frac{d-s}{2}} V(x) = \mathcal{F}^{-1}\Big(\|\xi\|^{d-s} \mathcal{F}(V)(\xi)\Big)(x).$$

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Proposition (Kwaśnicki '15; Chafaï, Saff, Womersley '22)

For $0 < s < d$, suppose that μ_{eq} minimizes $I_{W_s,V}$. Then on $\operatorname{supp}(\mu_{eq})^\circ$,
$d\mu_{eq}(x) = -(-\Delta)^{\frac{d-s}{2}} V(x) dx.$

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Corollary

If $0 < s < d$ and $(-\Delta)^{\frac{d-s}{2}} V(x) > 0$, then $x \notin \operatorname{supp}(\mu_{eq})^\circ$. If $(-\Delta)^{\frac{d-s}{2}} V(x) > 0$ everywhere, then $\operatorname{supp}(\mu_{eq})^\circ = \emptyset$.

Harmonic Case: $s = d - 2$

Corollary

For $0 < s = d - 2$, *on* $\mathrm{supp}(\mu)^\circ$, $d\mu_{eq}(x) = \Delta V(x)dx$.

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Theorem (Mhaskar, Saff '92; López-García '10)

Suppose $0 \leq s = d - 2$, $V(x) = g(\|x\|)$, g' *is absolutely continuous and nonnegative on* $(0, \infty)$, *and* $r^{d-1}g'(r)$ *is increasing with* $\lim_{r\to\infty} r^{d-1}g'(r) > 1$. *If* R *is the solution of* $R^{d-1}g'(R) = 1$, *then* $I_{W_{d-2},V}$ *is minimized by*

$$d\mu_{eq}(x) = \left(r^{d-1}g'(r)\right)' \mathbb{1}_{[0,R]}(r)drd\sigma(\theta).$$

Harmonic Case: $s = d - 2$

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For $0 < s = d - 2$, on $\text{supp}(\mu)^\circ$, $d\mu_{eq}(x) = \Delta V(x)dx$.

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Suppose $0 \leq s = d - 2$, $V(x) = g(\|x\|)$, g' is absolutely continuous and nonnegative on $(0, \infty)$, and $r^{d-1}g'(r)$ is increasing with $\lim_{r\to\infty} r^{d-1}g'(r) > 1$. If R is the solution of $R^{d-1}g'(R) = 1$, then $I_{W_{d-2},V}$ is minimized by

$$d\mu_{eq}(x) = \left(r^{d-1}g'(r)\right)' \mathbb{1}_{[0,R]}(r)drd\sigma(\theta).$$

Corollary

If $0 \leq s = d - 2$, $\alpha > 0$, $\gamma > 0$, and $R = \gamma^{-\frac{1}{d+\alpha-2}}$ then the minimizer of $I_{W_{d-2},V_{\alpha,\gamma}}$ is given by

$$d\mu_{eq}(x) = \gamma(d+\alpha-2)r^{d+a-3}\mathbb{1}_{[0,R]}(r)drd\sigma(\theta) = C_{d,\alpha,\gamma}\|x\|^{\alpha-2}\mathbb{1}_{B(0,R)}(x)dx.$$

The Next "Nice" Case: $s = d - 4$

Corollary

For $0 < s = d - 4$, *on* $\mathrm{supp}(\mu)^\circ$, $d\mu_{eq}(x) = -\Delta^2 V(x)dx$.

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Theorem (Chafaï, Saff, Womersley ’22)

If $0 \leq s = d - 4$, $0 < \alpha < 2$, $\gamma > 0$, *and* $R = \left(\frac{2}{\gamma(d+\alpha-2)}\right)^{\frac{1}{d+\alpha-4}}$ *then the unique minimizer of*

$$I_{W_{d-4},V_{\alpha,\gamma}}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{d-4}\|x-y\|^{4-d} + \gamma\frac{\|x\|^\alpha}{\alpha} + \gamma\frac{\|y\|^\alpha}{\alpha} d\mu(x)d\mu(y)$$

is given by

$$d\mu_{eq}(x) = \frac{d+\alpha-4}{d-2}d\sigma_R(x) + \frac{\gamma(2-\alpha)(d+\alpha-2)(d+\alpha-4)}{2(d-2)} r^{d+\alpha-5}\mathbb{1}_{[0,R]}(r)drd\sigma(\theta).$$

Quadratic External Field

Theorem (Various authors; Chafaï, M., Saff, Vu, Womersley)

For $d \geq 2$, $\alpha = 2$, *and* $\gamma > 0$,

- *if* $d - 4 < s < d$, *then*

$$d\mu_{eq}(x) = A_{s,d,R}(R^2 - \|x\|^2)^{1-\frac{d-s}{2}} \mathbb{1}_{B(0,R)}(x)dx,$$

- *if* $-2 < s \leq d - 4$, *then* $\mu_{eq} = \sigma_R$.

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- *if $-2 < s \leq d - 4$, then $\mu_{eq} = \sigma_R$.*

Theorem (Classical; Riesz, 1938; Björck, 1956)

Let $V_{B(0,R)} = \begin{cases} 0 & \|x\| \leq R \\ \infty & \|x\| > R \end{cases}$. If μ_{eq} is the minimizer of $I_{W_s, V_{B(0,R)}}$, then

- *if $d - 2 < s < d$,*

$$d\mu(x) = C_{d,s,R}\left(R^2 - \|x\|^2\right)^{-\frac{d-s}{2}} \mathbb{1}_{B(0,R)}(x)dx.$$

- *if $-2 < s \leq d - 2$, $\mu = \sigma_R$*

For a compact set $A \subset \mathbb{R}^d$, let $V_A(x) = \begin{cases} 0s & x \in A \\ \infty & x \notin A \end{cases}$, so the energy integral becomes

$$I_{W_s,V_A}(\mu) = \int_A \int_A \frac{1}{s} \|x - y\|^{-s} d\mu(x) d\mu(y).$$

Riesz Energy on Compacta

For a compact set $A \subset \mathbb{R}^d$, let $V_A(x) = \begin{cases} 0s & x \in A \\ \infty & x \notin A \end{cases}$, so the energy integral becomes

$$I_{W_s,V_A}(\mu) = \int_A \int_A \frac{1}{s} \|x-y\|^{-s} d\mu(x) d\mu(y).$$

Theorem

Suppose there exists some $\mu \in \mathcal{P}(A)$ with

$$\int_A \int_A W_s(\|x-y\|) d\mu(x) d\mu(y) < \infty.$$

Then $\dim(A) > s$.

Riesz Energy on Compacta

Theorem (Wallins,'52)

If $d-2<s<d$, *A is convex, and* $A^\circ \neq 0$, *then* $\text{supp}(\mu_{eq}) = A$.

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Theorem

If $-2<s\leq d-2$ *and A is convex, then* $\mathrm{supp}(\mu_{eq})\subseteq\partial A$.

Theorem (Björck, 1956)

If $s<-1$, *then* $\mathrm{supp}(\mu_{eq})$ *consists only of the extreme points of the convex hull of A.*

If $s<-2$, *and* μ *is a minimizer of* I_{W_s,V_A}, *then* $\#\,\mathrm{supp}(\mu)\leq d+1$.

Theorem (Wallins,'52)

If $d - 2 < s < d$, A is convex, and $A^\circ \neq 0$, then $\mathrm{supp}(\mu_{eq}) = A$.

Theorem

If $-2 < s \leq d - 2$ and A is convex, then $\mathrm{supp}(\mu_{eq}) \subseteq \partial A$.

Theorem (Björck, 1956)

If $s < -1$, then $\mathrm{supp}(\mu_{eq})$ consists only of the extreme points of the convex hull of A.

If $s < -2$, and μ is a minimizer of I_{W_s,V_A}, then $\# \mathrm{supp}(\mu) \leq d + 1$.

For the cube $A = [0, 1]^d$, this means that for $s < -1$, μ_{eq} has finite support.

Riesz Energy on the Ball

Consider $I_{W_s,V_A}(\mu) = \int_A \int_A \frac{1}{s}\|x-y\|^{-s} d\mu(x)d\mu(y)$ for $A = B(0,R)$.

Theorem (Riesz, 1938; Björck, 1956)

If μ is a minimizer of $I_{W_s,V_{B(0,R)}}$, then

- *if $d-2 < s < d$,*

$$d\mu(x) = C_{d,s,R}\left(R^2 - \|x\|^2\right)^{-\frac{d-s}{2}} \mathbb{1}_{B(0,R)}(x)dx.$$

- *if $-2 < s \leq d-2$, $\mu = \sigma_R$*
- *if $s = -2$, μ is supported on $R\mathbb{S}^{d-1}$ and has center of mass at the origin*
- *If $s < -2$, $\mu = \frac{1}{2}(\delta_p + \delta_{-p})$ for some $p \in R\mathbb{S}^{d-1}$.*

Acts as the limiting case $\alpha \to \infty$ of

$$I_{W_s,V_{\alpha,\gamma}}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1}{s}\|x-y\|^{-s} + \gamma\frac{\|x\|^\alpha}{\alpha} + \gamma\frac{\|y\|^\alpha}{\alpha}\right)d\mu(x)d\mu(y).$$

Some Open Problems

- What are sufficient conditions for a general V that will guarantee a compactly supported minimizer of $I_{W_s,V}$? Are there any necessary and sufficient conditions?
- What are the explicit minimizers for other combinations of s and α?
    - When is the support a ball with disjoint concentric spheres?
    - When is it an annulus?
    - When are there singular components, and when are there not?
    - What do we get for $s \leq -2$?
- Can we find similar bounds on the dimension of minimizers of $I_{W_s,V}$ as with repulsive attractive kernels?
    - $d-2 \leq s < d$: $\dim(\text{supp}(\mu_{eq})) = d$
    - $-2 < s < d-2$: $\dim(\text{supp}(\mu_{eq})) \geq s+2$
    - $s < -2$: $\text{supp}(\mu_{eq})$ is finite.

  Can we improve on these bounds? Can we get upper bounds based on strength of attraction?

Some Open Problems

- How might a lack of smoothness or continuity in $V(x)$ affect the dimension of a minimizer?

- For electrons around a positive source at the origin, we would have $V(x) = -\gamma \frac{1}{||x||}$. However, Potential Theoretic methods generally require that $W_s(x-y) + V(x) + V(y)$ be bounded from below on every compact subset of $\mathbb{R}^d$. How can we handle external fields with negative singularities?

  Idea: If $V(x) = -\infty$ for all $x \in A \subset \mathbb{R}^d$, let $Q_{A,r} = \cup_{x \in A} B(r,x)$. We can use Potential Theoretic methods to find a minimizing measure $\mu_{eq,r}$ of $I_{W_s,V}$ on $\Omega_{A,r} = \mathbb{R}^d \setminus Q_{A,r}$. By then taking $r \to \infty$, we should arrive at the optimal measure μ_{eq} for $I_{W_s,V}$ on $\mathbb{R}^d$.

Thank you!

The research in this presentation is in collaboration with Djalil Chafaï, Edward Saff, Minh Vu, and Robert Womersley and is supported in part by the NSF Mathematical Sciences Postdoctoral Research Fellowship.