Stability and long-time behavior of a heavy rigid body with a cavity completely filled with a viscous liquid

Giusy Mazzone

Department of Mathematics

VANDERBILT
UNIVERSITY

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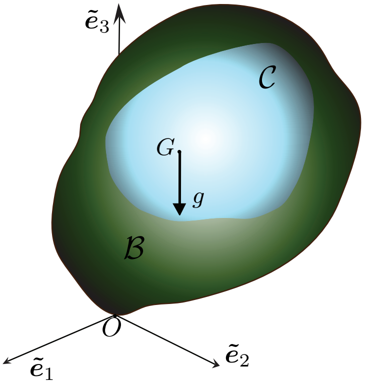
Motions of a liquid-filled rigid body about a fixed point

Consider a rigid body $\mathcal{B}$ with a cavity, $\mathcal{C}$, completely filled with a viscous liquid

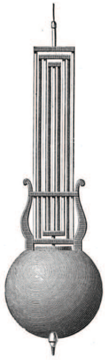


We have investigated asymptotic behavior and stability of the coupled system when it moves around a fixed point under the action of gravity:

- *motions of a liquid-filled physical pendulum*;
- *motions of a liquid-filled spherical pendulum*;
- *motions of a liquid-filled spinning top*.

Examples

$\mathcal{B}$ moves while keeping constant the distance between its center of mass and a fixed point O.



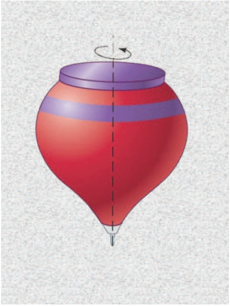


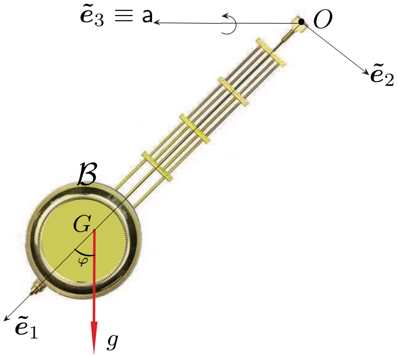
Figure: Physical Pendulum (left), Spherical Pendulum (center), Spinning Top (right).

The physical pendulum

A physical pendulum[1] is a heavy rigid body, $\mathcal{B}$, constrained to rotate around a horizontal axis, a, so that its center of mass G satisfies the following properties:

(i) the distance, ℓ, between G and its orthogonal projection O on a (point of suspension), does not depend on time,

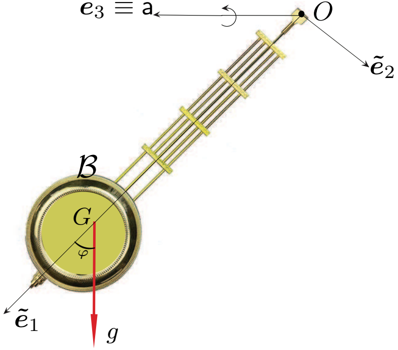
(ii) G always moves in a plane orthogonal to a.



[1] No liquid

The physical pendulum

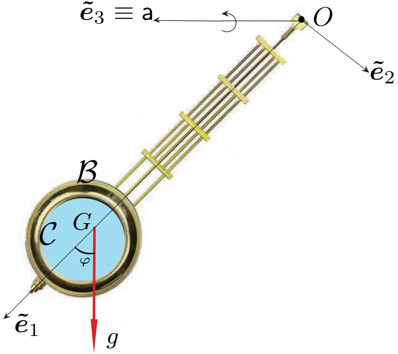
In absence of friction, the generic motion of $\mathcal{B}$ is a nonlinear oscillation: motions of "small amplitude" around the lowest position of G are undamped oscillations with frequency $\sqrt{mg\ell/I}$, where g is the acceleration of gravity and m and I represent the mass of $\mathcal{B}$ and its moment of inertia around a, respectively.



The physical pendulum

Question

How does the dynamics of this physical system change if the cavity is completely filled by a viscous incompressible fluid (liquid)? In other words, how the long-time behavior and the stability of the couple system is effected?



Applications of a liquid-filled heavy solid

In space engineering:

- study of the motion of fuel within the tank; [1]
- tube dampers filled with a viscous liquid are used to suppress oscillations in spacecraft and artificial satellites. [2]

[1] Abramson, H. N. (1966) *Dynamic behavior of liquids in moving containers with applications to propellants in space vehicle fuel tanks*. NASA-SP-106.

[2] Bhuta, P.G. & Koval, L.R. (1966) *A viscous ring damper for a freely precessing satellite*. Intern. J. Mech. Sci. **8** 5.

Idea and preliminary results

Idea:

The liquid has a stabilizing effect on the motion of the solid: after an initial "chaotic" motion, whose duration, t_0, depends on the "size" of the initial data as well as on the relevant physical parameters involved (viscosity and density of the liquid, mass distribution of the rigid body, etc.), the coupled system reaches a more orderly configuration (corresponding to an equilibrium).

Previous literature concerning the motions of a rigid body having a cavity entirely filled with an ideal, irrotational, incompressible liquid

- G. Stokes (1880),
- N. Y. Zhukovskii (1885),
- S. S. Hough (1895),
- H. Poincaré (1910),
- S. L. Sobolev (1960).

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Previous literature concerning the stability of motion of a rigid body with a cavity partially or entirely filled by ideal and viscous liquids

- V. V. Rumyantsev (1960),
- F. L. Chernousko (1972),
- E. P. Smirnova (1974),
- A. A. Lyashenko (1993),
- N. D. Kopachevsky and S. G. Krein (2000)
- . . . .

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More recent results concerning inertial motions

- A.L. Silvestre and T. Takahashi, *On the Motion of a Rigid Body with a Cavity Filled with a Viscous Liquid*, Proc. Roy. Soc. Edinburgh (2012)
- *A mathematical analysis of the motion of a rigid body with a cavity containing a newtonian fluid*. Ph.D. thesis, Università del Salento (2012)

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This stabilizing effect has been rigorously proved in the case of inertial motions

- (with J. Prüss and G. Simonett) *Stability properties and asymptotic behavior of a fluid-filled rigid body in critical spaces*, in preparation (2017)
- G. P. Galdi, *Stability of permanent rotations and long-time behavior of inertial motions of a rigid body with an interior liquid-filled cavity*, arXiv (2017)
- *On the dynamics of a rigid body with cavities completely filled by a viscous liquid*. Ph.D. thesis, University of Pittsburgh (2016)
- (with K. Disser, G. P. Galdi and P. Zunino) *Inertial motions of a rigid body with a cavity filled with a viscous liquid*, Arch. Rational Mech. Anal., **221** (1), (2016)
- (with G. P. Galdi, and P. Zunino) *Inertial Motions of a Rigid Body with a Cavity Filled with a Viscous Liquid*, arXiv:1405.6596 (2014)

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- (with G. P. Galdi) *Stability and Long-Time Behavior of a Pendulum with an Interior Cavity Filled with a Viscous Liquid*. Submitted (2017).
- (with G.P. Galdi and M. Mohebbi) *On the motion of a liquid-filled heavy body around a fixed point*. Accepted in Quart. Appl. Math. (2017)
- (with G.P. Galdi) *On the motion of a pendulum with a cavity entirely filled with a viscous liquid*. Ch. in "Recent progress in the theory of the Euler and Navier-Stokes Equations", London Math. Soc. Lecture Note Ser., 430, Cambridge Univ. Press, 2016
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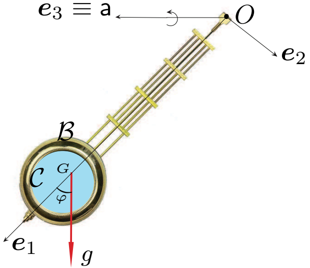
Motion of a liquid-filled pendulum

Let $\mathcal{S}$ be the coupled system constituted by a rigid body, $\mathcal{B}$, with an interior cavity, $\mathcal{C}$ (assumed to be a domain of $\mathbb{R}^3$ of class C^2), entirely filled with a viscous liquid. Suppose that

- $\mathcal{B}$ is constrained to move (without friction) around a horizontal axis a,
- the center of mass G of $\mathcal{S}$ belongs to a fixed vertical plane orthogonal to a,
- the distance from G to its orthogonal projection, O, on a is kept constant.



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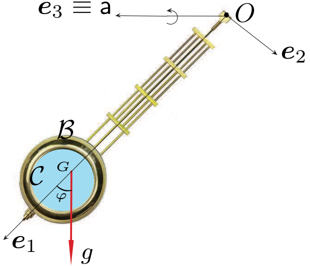

- $\mathcal{F} \equiv \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a *moving frame* attached to $\mathcal{B}$,
- the angular velocity is: $\omega(t)\boldsymbol{e}_3$,
- the gravity is given by $g\boldsymbol{\chi} \equiv g(\cos\varphi, -\sin\varphi, 0)$ (it is a time-dependent vector).

Equations of motion in the moving frame

The motion of $\mathcal{S}$ in $\mathcal{F}$ is governed by the following set of equations

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{v}=0 \\ \rho\left(\boldsymbol{v}_t+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\dot{\omega}\boldsymbol{e}_3\times\boldsymbol{x}+2\omega\,\boldsymbol{e}_3\times\boldsymbol{v}\right)=\mu\Delta\boldsymbol{v}-\nabla p \end{aligned}\right\}\quad \text{in } \mathcal{C}\times\mathbb{R}_+,$$

$$\boldsymbol{v}(x,t)=\mathbf{0} \qquad \text{on } \partial\mathcal{C}\times\mathbb{R}_+,$$

$$\mathsf{C}(\dot{\omega}-\dot{a})=\beta^2\chi_2 \qquad \text{in } \mathbb{R}_+,$$

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$$\dot{\boldsymbol{\chi}} + \omega\,\boldsymbol{e}_3 \times \boldsymbol{\chi} = \mathbf{0} \qquad \text{in } \mathbb{R}_+,$$

Here, $\boldsymbol{v}$ is the fluid velocity *relative* to $\mathcal{B}$ and $p := \tilde{p}/\rho - g\boldsymbol{\chi}\cdot\boldsymbol{x}$ is its *modified pressure*; ρ and μ are the fluid density and shear viscosity coefficient. The first two equations are the so-called Navier-Stokes equations, and describe the motion of the liquid subject to no-slip boundary conditions (Dirichlet boundary conditions).

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Here, C is the moment of inertia of $\mathcal{S}$ with respect to a,

$$a:=-\frac{\rho}{\mathsf{C}}\,\boldsymbol{e}_3\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{v}\,,dV \qquad \text{and} \qquad \beta^2=M\,g\,|\,\overrightarrow{OG}\,|,$$

with M mass of $\mathcal{S}$. This equation describe the balance of total angular momentum of $\mathcal{S}$ with respect to O.

Equations of motion in the moving frame

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Here, $\chi_1\equiv\cos\varphi$ and $\chi_2\equiv-\sin\varphi$. This equation describe the time-evolution of the direction of the gravity in the moving frame.

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$$\dot{\boldsymbol{\chi}} + \omega\,\boldsymbol{e}_3\times\boldsymbol{\chi} = \boldsymbol{0} \qquad \text{in } \mathbb{R}_+.$$

We can formally obtain the following energy balance:

$$\frac{d}{dt}\left[\mathcal{E}+\mathcal{U}\right] + \mu\|\nabla\boldsymbol{v}(t)\|_2^2 = 0,$$

where $\mathcal{U} := -\beta^2\chi_1$ (potential energy) and

$$\mathcal{E} := \frac{1}{2}\left[\rho\,\|\boldsymbol{v}\|_2^2 - \mathsf{C}\,a^2 + \mathsf{C}\,(\omega-a)^2\right] \quad \text{(kinetic energy)}.$$

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Moreover, we have the constraint $|\boldsymbol{\chi}(t)| = 1$ at all times.

Steady-states solutions

Steady-state solutions can be found by imposing that $(\boldsymbol{v}_t, \dot{\omega}, \dot{\boldsymbol{\chi}}) = \mathbf{0}$:

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{v} = 0 \\ \rho\left(\cancelto{0}{\boldsymbol{v}_t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \cancelto{0}{\dot{\omega}\boldsymbol{e}_3\times\boldsymbol{x}} + 2\omega\,\boldsymbol{e}_3\times\boldsymbol{v}\right) = \mu\Delta\boldsymbol{v} - \nabla p \end{aligned}\right\} \quad \text{in } \mathcal{C},$$

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$$\beta^2 \chi_2 = 0\,,$$

$$\omega\,\boldsymbol{e}_3 \times \boldsymbol{\chi} = \mathbf{0}\,.$$

The previous system then has only two solutions given by

$$\mathsf{s}_0^{\pm} := (\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}, \omega \equiv 0, \boldsymbol{\chi} = \pm \boldsymbol{e}_1)\,.$$

They represent the equilibrium configurations where $\mathcal{S}$ is at rest with G in its lowest (s_0^+) or highest (s_0^-) position.

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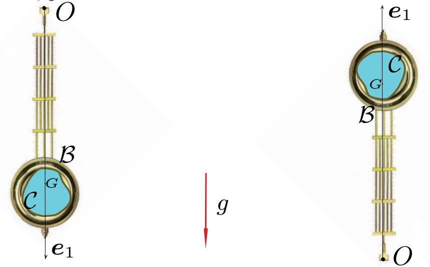

Figure: Only two possible equilibrium configurations where $\mathcal{S}$ is at rest with G in its lowest (s_0^+, left figure) or highest (s_0^-, right figure) position.

Asymptotic Stability of the Equilibrium Configurations

Consider the "perturbed motion" around $\mathsf{s}_0^{\pm} = (\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}, \omega \equiv 0, \boldsymbol{\chi} = \pm \boldsymbol{e}_1)$

$$(\boldsymbol{v}, p, \omega, \boldsymbol{\chi} := \boldsymbol{\gamma} \pm \boldsymbol{e}_1)\,, \quad |\boldsymbol{\gamma} \pm \boldsymbol{e}_1| = 1\,.$$

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The "perturbation" $(\boldsymbol{v}, p, \omega, \boldsymbol{\gamma})$ has to satisfy the following equations

$$\left.\begin{aligned} &\nabla \cdot \boldsymbol{v} = 0 \\ &\rho\,(\boldsymbol{v}_t + \dot{\omega}\boldsymbol{e}_3 \times \boldsymbol{x} + 2\omega\,\boldsymbol{e}_3 \times \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \mu \Delta \boldsymbol{v} + \nabla p = \mathbf{0} \end{aligned}\right\} \ \text{in } \mathcal{C} \times \mathbb{R}_+,$$

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$$\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2 \gamma_2 , \quad \dot{\boldsymbol{\gamma}} + \omega\,\boldsymbol{e}_3 \times \boldsymbol{\gamma}_0 + \omega\,\boldsymbol{e}_3 \times \boldsymbol{\gamma} = \mathbf{0} ,$$

with $\boldsymbol{\gamma}_0 := \xi\,\boldsymbol{e}_1$, $\xi = \pm 1$.

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with $\boldsymbol{\gamma}_0 := \xi\,\boldsymbol{e}_1\,, \quad \xi = \pm 1\,.$

The idea is to write the previous system of equations as an evolution problem

$$\frac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \mathbf{0}\,, \quad \boldsymbol{u}(0) \in H$$

on an appropriate Hilbert space H.

The evolution problem

Let us consider the Hilbert space

$$H := \left\{ \boldsymbol{u} := (\boldsymbol{v}, \omega, \boldsymbol{\gamma})^T : \boldsymbol{u} \in L^2_\sigma(\mathcal{C}) \oplus \mathbb{R} \oplus \mathbb{R}^2 \right\} ,$$

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$$L^2_\sigma(\mathcal{C}) := \left\{ \boldsymbol{v} \in L^2(\mathcal{C}) : \ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \mathcal{C} , \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\mathcal{C}} = 0 \right\} .$$

Moreover, the Helmholtz-Weyl decomposition holds:

$$L^2(\mathcal{C}) = L^2_\sigma(\mathcal{C}) \oplus G(\mathcal{C}),$$

where $G(\mathcal{C}) := \{ \boldsymbol{w} \in L^2(\mathcal{C}) : \ \boldsymbol{w} = \nabla \pi \quad \text{for some } \pi \in W^{1,2}_{loc}(\mathcal{C}) \}$. Then, in the Navier-Stokes equations

$$\left.\begin{array}{l} \cancel{\nabla \cdot \boldsymbol{v} = 0} \\ \rho \left[\boldsymbol{v}_t + \mathsf{P}(\dot{\omega} \boldsymbol{e}_3 \times \boldsymbol{x} + 2\omega\, \boldsymbol{e}_3 \times \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) \right] - \mu \mathsf{P} \Delta \boldsymbol{v} + \cancel{\nabla p} = \boldsymbol{0} \end{array} \right\} \text{ in } \mathcal{C} \times \mathbb{R}_+,$$

where $\mathsf{P} : L^2(\mathcal{C}) \mapsto L^2_\sigma(\mathcal{C})$ is the Helmholtz-Weyl projector.

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endowed with the inner product

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle := \int_{\mathcal{C}} \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \, dV + \omega_1 \, \omega_2 + \boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2 \, ,$$

and corresponding norm

$$\|\boldsymbol{u}\| := \langle \boldsymbol{u}, \boldsymbol{u} \rangle^{\frac{1}{2}} \, .$$

The evolution problem

Let us consider the Hilbert space

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and introduce the following operators

$$\boldsymbol{I} : \boldsymbol{u} \in H \mapsto \boldsymbol{I}\boldsymbol{u} := (\rho\, \boldsymbol{v} + \mathsf{P}[\rho\, \omega \boldsymbol{e}_3 \times \boldsymbol{x}], \mathsf{C}(\omega - a), \boldsymbol{\gamma})^T \in H$$

$$\tilde{\boldsymbol{A}} : \boldsymbol{u} \in D(\tilde{\boldsymbol{A}}) \mapsto \tilde{\boldsymbol{A}}\boldsymbol{u} := (-\mu\, \mathrm{P}\Delta \boldsymbol{u}, \omega, \boldsymbol{\gamma})^T \in H$$

$$D(\tilde{\boldsymbol{A}}) := L^2_\sigma(\mathcal{C}) \cap W^{1,2}_0(\mathcal{C}) \cap W^{2,2}(\mathcal{C}) \oplus \mathbb{R} \oplus \mathbb{R}^2 \subset H$$

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is bounded, invertible and symmetric

$$\tilde{\boldsymbol{A}} : \boldsymbol{u} \in D(\tilde{\boldsymbol{A}}) \mapsto \tilde{\boldsymbol{A}}\boldsymbol{u} := (-\mu\, \mathrm{P}\Delta \boldsymbol{u}, \omega, \boldsymbol{\gamma})^T \in H$$

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The equations for the perturbation fileds can be written as the following evolution equation in the space H

$$\frac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0}\,, \quad \boldsymbol{u}(0) \in H$$

where $\boldsymbol{L} := \boldsymbol{I}^{-1}(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}})$, $D(\boldsymbol{L}) = D(\tilde{\boldsymbol{A}})$.

Generalized linearization principles

Theorem [a]

[a] Kirchgässner, K.& Kielhöfer, H.,(1973), see also Henry, D., (1981)

1. Let $\boldsymbol{A}$ be a linear, sectorial operator with compact inverse and Re$[\sigma(\boldsymbol{A})] > 0$.
2. For $\alpha \in [0,1]$, set $X_\alpha = \{\boldsymbol{u} \in H : \ \|\boldsymbol{u}\|_\alpha := \|\boldsymbol{A}^\alpha \boldsymbol{u}\| < \infty\}$; $X_0 \equiv H$.
3. Let the operator B be a bounded linear map from X_α to H.
4. Assume that the nonlinear operator $\boldsymbol{N}$ satisfies

$$\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c\,\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha\,, \text{ all } \boldsymbol{u}_1, \boldsymbol{u}_2 \text{ in a neighborhood of } \boldsymbol{0} \in H\,.$$

5. Set $\boldsymbol{L} = \boldsymbol{A} + \boldsymbol{B}$, and suppose that Re$[\sigma(\boldsymbol{L})] \subset \{\lambda \in \mathbb{C} : \ \text{Re}\lambda > \beta\}$, for some $\beta > 0$.
6. Let $\boldsymbol{u}$ be a solution to $\dfrac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0}\,, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in H$.

Then, there exists $\rho > 0$ and $M \geq 1$ such that if $\|\boldsymbol{u}_0\|_\alpha \leq \rho$, one has

$$\|\boldsymbol{u}(t)\|_\alpha \leq M e^{-\beta t} \|\boldsymbol{u}_0\|_\alpha, \qquad \text{for all } t \geq 0.$$

Generalized linearization principles (continued)

Unfortunately, for the problem at our hand, the hypothesis

$$\text{Re}[\sigma(\boldsymbol{L})] \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda > \beta\}, \qquad \text{for some } \beta > 0.$$

is NOT satisfied! In fact, our nonlinear evolution problems has a slow (local) center manifold, that is, the spectrum of the relevant linear (time-independent) operator, $\boldsymbol{L}$, is discrete and $\sigma(\boldsymbol{L}) \cap i\mathbb{R} = \{0\}$.

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$\lambda = 0$ is an eigenvalue of $\boldsymbol{L}$.

The equation $\boldsymbol{L}\boldsymbol{u} = \boldsymbol{0}$ in H is equivalent to the following system of equations:

$$\begin{aligned} &-\mu\,\Delta\boldsymbol{v} + \nabla p = \boldsymbol{0}\,,\ \nabla\cdot\boldsymbol{v} = 0\,,\ \ \boldsymbol{v}|_{\partial\mathcal{C}} = \boldsymbol{0} \\ &\beta^2\gamma_2 = 0\,,\ \ \omega\,\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{0}\,, \end{aligned}$$

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Remark

$\mathsf{N}[\boldsymbol{L}] = span\{\boldsymbol{e}_1\}$, so $\dim \mathsf{N}[\boldsymbol{L}] = 1$.

Generalized linearization principles (continued)

Theorem (Stability)[a]

[a] G. P. Galdi, & G. M. (2017)

Let the following hypotheses be satisfied.

1. The linear operator $\boldsymbol{L}$ is Fredholm of index 0, sectorial, has a discrete spectrum with $\mathrm{Re}[\sigma(\boldsymbol{L}) \setminus \{0\}] > 0$.
2. $\dim \mathsf{N}[\boldsymbol{L}] = m \geq 1$.
3. $\mathsf{N}[\boldsymbol{L}] \cap \mathsf{R}[\boldsymbol{L}] = \{\boldsymbol{0}\}$.
4. $\sigma(\boldsymbol{L}) \cap \{i\mathbb{R}\} = \{0\}$.
5. $\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c_1 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha$, for all $\boldsymbol{u}_1, \boldsymbol{u}_2$ in a neighborhood of $\boldsymbol{0} \in H$.
6. For every $\boldsymbol{u} \in H$, $\|\boldsymbol{N}(\boldsymbol{u})\| \leq c_2 \left[(\|\boldsymbol{u}^{(0)}\| + \|\boldsymbol{u}^{(1)}\|^{\kappa_1}) \|\boldsymbol{u}^{(1)}\|^{\kappa_2} + \|\boldsymbol{u}^{(1)}\|_\alpha^{\kappa_3} \right]$, $\kappa_1, \kappa_2 \geq 1,\ \kappa_3 > 1$, where $\boldsymbol{u}^{(0)} = \mathcal{Q}(\boldsymbol{u})$ and $\boldsymbol{u}^{(1)} = \mathcal{P}(\boldsymbol{u})$, with $\mathcal{Q}$ and $\mathcal{P}$ the spectral projections according to $\sigma_0(\boldsymbol{L}) = \{0\}$ and $\sigma_1(\boldsymbol{L}) = \sigma(\boldsymbol{L}) \setminus \{0\}$.

Then, there exists $\rho_0 > 0$ such that if $\|\boldsymbol{u}(0)\|_\alpha < \rho_0$, there is a unique corresponding solution $\boldsymbol{u} = \boldsymbol{u}(t) \in C([0,T]; X_\alpha) \cap C((0,T]; X_1) \cap C^1((0,T]; H)$, all $T > 0$, to

$$\frac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0} \qquad \text{for all } t > 0.$$

Generalized linearization principles (continued)

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The solution $\boldsymbol{u} = \mathbf{0}$ is *stable* in X_α:

(a) For any $\varepsilon > 0$ there is $\delta > 0$ such that $\|\boldsymbol{u}(0)\|_\alpha < \delta \quad \Rightarrow \quad \sup_{t \geq 0} \|\boldsymbol{u}(t)\|_\alpha < \varepsilon$.

Generalized linearization principles (continued)

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Let the following hypotheses be satisfied.

1. The linear operator $\boldsymbol{L}$ is Fredholm of index 0, sectorial, has a discrete spectrum with $\mathrm{Re}[\sigma(\boldsymbol{L}) \setminus \{0\}] > 0$.
2. $\dim \mathsf{N}[\boldsymbol{L}] = m \geq 1$.
3. $\mathsf{N}[\boldsymbol{L}] \cap \mathsf{R}[\boldsymbol{L}] = \{\mathbf{0}\}$.
4. $\sigma(\boldsymbol{L}) \cap \{i\mathbb{R}\} = \{0\}$.
5. $\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c_1 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha$, for all $\boldsymbol{u}_1, \boldsymbol{u}_2$ in a neighborhood of $\mathbf{0} \in H$.
6. For every $\boldsymbol{u} \in H$, $\|\boldsymbol{N}(\boldsymbol{u})\| \leq c_2 \left[(\|\boldsymbol{u}^{(0)}\| + \|\boldsymbol{u}^{(1)}\|^{\kappa_1}) \|\boldsymbol{u}^{(1)}\|^{\kappa_2} + \|\boldsymbol{u}^{(1)}\|_\alpha^{\kappa_3} \right]$, $\kappa_1, \kappa_2 \geq 1,\ \kappa_3 > 1$, where $\boldsymbol{u}^{(0)} = \mathcal{Q}(\boldsymbol{u})$ and $\boldsymbol{u}^{(1)} = \mathcal{P}(\boldsymbol{u})$, with $\mathcal{Q}$ and $\mathcal{P}$ the spectral projections according to $\sigma_0(\boldsymbol{L}) = \{0\}$ and $\sigma_1(\boldsymbol{L}) = \sigma(\boldsymbol{L}) \setminus \{0\}$.

The solution $\boldsymbol{u}$ converges in X_α *exponentially fast* to a point in $\mathsf{N}[\boldsymbol{L}]$:

(b) there are $\eta, c, \kappa > 0$ such that $\|\boldsymbol{u}(0)\|_\alpha < \eta \quad \Rightarrow$ there exists $\bar{\boldsymbol{u}} \in \mathsf{N}[\boldsymbol{L}]$ such that $\|\boldsymbol{u}(t) - \bar{\boldsymbol{u}}\|_\alpha \leq c \|\boldsymbol{u}^{(1)}(0)\|_\alpha e^{-\kappa t}$, all $t > 0$.

A "visual" example in finite dimensions

Consider the following system of nonlinear 1st-order ODEs

$$\begin{cases} \dfrac{dx}{dt} = -x(x-1) \\ \dfrac{dy}{dt} = x-1. \end{cases}$$

There is a 1-dimensional manifold of equilibria $\mathsf{E} = \{(x,y) \in \mathbb{R}^2 : \ x = 1\}$.

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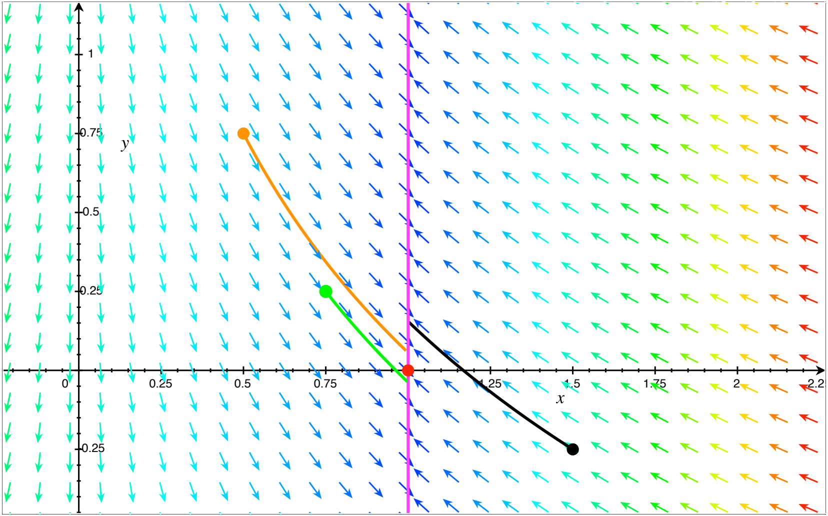
$$\begin{cases} \dfrac{dx}{dt} = -x(x-1) \\ \dfrac{dy}{dt} = x-1. \end{cases} \quad \Leftrightarrow \quad \begin{array}{l} \dfrac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0}, \quad \boldsymbol{u} := (x-1, y)^T, \\ \boldsymbol{L} := \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \ \boldsymbol{N}(\boldsymbol{u}) := \begin{bmatrix} (x-1)^2 \\ 0 \end{bmatrix} \end{array}$$

There is a 1-dimensional manifold of equilibria $\mathsf{E} = \{(x,y) \in \mathbb{R}^2 : \ x = 1\}$.
Let $\boldsymbol{L}$ be the linearization around the equilibrium $(x_*, y_*) \equiv (1,0)$. Then, one notice the following properties.

1. $\det \boldsymbol{L} = 0$, so $\lambda = 0$ is an eigenvalue of $\boldsymbol{L}$. Moreover, $\sigma(\boldsymbol{L}) = \{0, 1\}$.
2. $\mathsf{N}[\boldsymbol{L}] = \mathsf{E}$, so $\dim \mathsf{N}[\boldsymbol{L}] = 1$.
3. $\mathsf{N}[\boldsymbol{L}] \cap \mathsf{R}[\boldsymbol{L}] = \{\boldsymbol{0}\}$.
4. $\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c_1 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|$ in a neighborhood of $\boldsymbol{0} \in H$.
5. $\|\boldsymbol{N}(\boldsymbol{u})\| = \|\boldsymbol{u}\|^2$.

A "visual" example in finite dimensions



Generalized linearization principles (continued)

Theorem (Instability)[a]

[a] D. Henry (1981), G. P. Galdi, & G. M. (2017)

Let the following hypotheses be satisfied.

1. The linear operator $\boldsymbol{L}$ is Fredholm of index 0, sectorial, has a discrete spectrum with $\mathrm{Re}[\sigma(\boldsymbol{L}) \setminus \{0\}] \cap (-\infty, 0) \neq \emptyset$.
2. $\dim \mathsf{N}[\boldsymbol{L}] = m \geq 1$.
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4. $\sigma(\boldsymbol{L}) \cap \{i\mathbb{R}\} = \{0\}$.
5. $\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c_1 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha$, for all $\boldsymbol{u}_1, \boldsymbol{u}_2$ in a neighborhood of $\mathbf{0} \in H$.
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Then, the solution $\boldsymbol{u} = \mathbf{0}$ is *unstable* in X_α:

(a) there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists an initial data $\|\boldsymbol{u}(0)\|_\alpha < \delta$ such that the corresponding solution satisfies $\sup_{t \geq 0} \|\boldsymbol{u}(t)\|_\alpha > \varepsilon$.

Some remarks

- Our stability (and instability) principles continue to hold if H is a Banach space.

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- Our stability (and instability) principles continue to hold if H is a Banach space.
- The stability results are in the spirit of the "generalized linearization principles" obtained by other authors (like Prüss, Simonett, Zacher (2009)), even though some of our assumptions and method of proof are different and specifically aimed at fluid-structure interaction problems.

Some remarks

- Our stability (and instability) principles continue to hold if H is a Banach space.
- The existence of a slow center manifold appears to be a basic characteristic of fluid-structure interaction problems. This is due to the fact that, for obvious physical reasons, the set of steady-state solutions does not reduce to a singleton, and may even form a continuum, either in absence or presence of a driving force.

Some remarks

- Our stability (and instability) principles continue to hold if H is a Banach space.
- The existence of a slow center manifold appears to be a basic characteristic of fluid-structure interaction problems.
- In the case of a liquid-filled spinning top (full three dimensional motion)

$$(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\gamma}) \text{ is a steady solution iff it satisfies } \begin{cases} \boldsymbol{v} \equiv \mathbf{0} \qquad \text{on } \mathcal{C}, \\ \boldsymbol{\omega} \times \boldsymbol{I} \cdot \boldsymbol{\omega} = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma}, \\ \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}. \end{cases}$$

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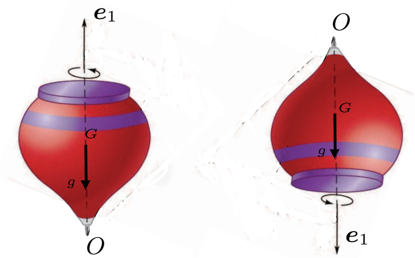

Figure: Permanent rotations around vertical axis.

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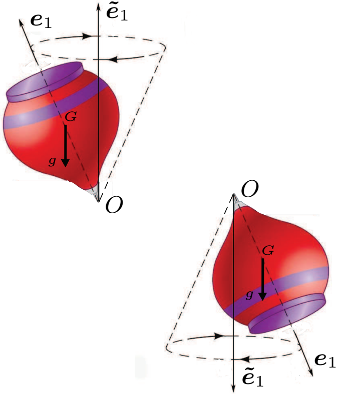

Figure: Steady precessions.

Main steps of the proof - Stability Theorem

Step 1. By classical results on semilinear evolution equations, there exists a local solution in some interval $(0, t_*)$ to

$$\frac{d\boldsymbol{u}}{dt} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0}, \qquad \boldsymbol{u}(0) \in H.$$

Moreover, either $t_* = \infty$ or $\|\boldsymbol{u}(t)\|_\alpha \to +\infty$ as $t \to t_*$.

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- The space H admits the decomposition $H = \mathsf{N}[\boldsymbol{L}] \oplus \mathsf{R}[\boldsymbol{L}]$.

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Moreover, either $t_* = \infty$ or $\|\boldsymbol{u}(t)\|_\alpha \to +\infty$ as $t \to t_*$.

Step 2. There exists $\rho_0 > 0$ such that if $\|\boldsymbol{u}(0)\|_\alpha < \rho_0$, then $t_* = +\infty$.

- The space H admits the decomposition $H = \mathsf{N}[\boldsymbol{L}] \oplus \mathsf{R}[\boldsymbol{L}]$.
- Let $\mathcal{Q}$ and $\mathcal{P}$ be the spectral projections according to the spectral sets $\{0\}$ and $\sigma(\boldsymbol{L}) \setminus \{0\}$, respectively. Then, $\mathsf{N}[\boldsymbol{L}] = \mathcal{Q}(H)$ and $\mathsf{R}[\boldsymbol{L}] = \mathcal{P}(H)$.

Main steps of the proof - Stability Theorem

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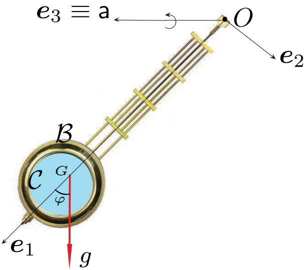
- Let $\mathcal{Q}$ and $\mathcal{P}$ be the spectral projections according to the spectral sets $\{0\}$ and $\sigma(\boldsymbol{L}) \setminus \{0\}$, respectively. Then, $\mathsf{N}[\boldsymbol{L}] = \mathcal{Q}(H)$ and $\mathsf{R}[\boldsymbol{L}] = \mathcal{P}(H)$.
- Set $\boldsymbol{L}_1 := \mathcal{P}\boldsymbol{L} = \boldsymbol{L}\mathcal{P}$, then $\mathrm{Re}[\sigma(\boldsymbol{L}_1)] > \gamma > 0$. Moreover, for every $\boldsymbol{v} \in H$, we write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$, $\boldsymbol{v}_0 \in \mathsf{N}[\boldsymbol{L}]$, $\boldsymbol{v}_1 \in \mathsf{R}[\boldsymbol{L}]$, and consider

$$\frac{d\boldsymbol{u}_1}{dt} + \boldsymbol{L}_1\boldsymbol{u}_1 = -\mathcal{P}\boldsymbol{N}(\boldsymbol{u}_0 + \boldsymbol{u}_1),$$

$$\frac{d\boldsymbol{u}_0}{dt} = -\mathcal{Q}\boldsymbol{N}(\boldsymbol{u}_0 + \boldsymbol{u}_1).$$

Asymptotic stability of a liquid-filled pendulum



Recall that we are perturbing the equations of motion around the equilibrium $\mathsf{s}_0^{\pm} = (\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}, \omega \equiv 0, \boldsymbol{\chi} = \pm \boldsymbol{e}_1)$, and we have written the equations for the perturbations as

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- Let $\boldsymbol{f} \equiv (0, 0, \sigma \boldsymbol{e}_1)^T \in \mathsf{N}[\boldsymbol{L}] \cap \mathsf{R}[\boldsymbol{L}]$,

$$-\mu \Delta \boldsymbol{v} + \nabla p = \mathbf{0}, \quad \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v}(x,t) = \mathbf{0} \ \text{ on } \partial\mathcal{C} \qquad \Rightarrow \boldsymbol{v} = \nabla p = \mathbf{0},$$

$$\beta^2 \gamma_2 = 0\,, \qquad \pm \omega\, \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \sigma \boldsymbol{e}_1\,. \qquad \Rightarrow \gamma_2 = 0, \qquad \sigma = 0.$$

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- $\sigma(\boldsymbol{L}) \cap \{i\mathbb{R}\} = \{0\}$.
- $\|\boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2)\| \leq c_1 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha$, for all $\boldsymbol{u}_1, \boldsymbol{u}_2$ in a neighborhood of $\mathbf{0} \in H$.
- For every $\boldsymbol{u} \in H$, $\|\boldsymbol{N}(\boldsymbol{u})\| \leq c_2 \left[(\|\boldsymbol{u}^{(0)}\| + \|\boldsymbol{u}^{(1)}\|^{\kappa_1})\|\boldsymbol{u}^{(1)}\|^{\kappa_2} + \|\boldsymbol{u}^{(1)}\|_\alpha^{\kappa_3}\right]$, $\kappa_1, \kappa_2 \geq 1, \ \kappa_3 > 1$.

Asymptotic stability of a liquid-filled pendulum (continued)

Lemma

1. *Consider* $\mathsf{s}_0^+ = (\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}, \omega \equiv 0, \boldsymbol{\chi} = \boldsymbol{e}_1)$, *then* $\mathrm{Re}[\sigma(\boldsymbol{L}) \setminus \{0\}] \subset (0, +\infty)$.
2. *Consider* $\mathsf{s}_0^- = (\boldsymbol{v} \equiv \nabla p \equiv \mathbf{0}, \omega \equiv 0, \boldsymbol{\chi} = -\boldsymbol{e}_1)$, *then* $\mathrm{Re}[\sigma(\boldsymbol{L}) \setminus \{0\}] \cap (-\infty, 0) \neq \emptyset$.

Figure: s_0^+ (left figure) and s_0^- (right figure).

Asymptotic stability of a liquid-filled pendulum (continued)

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Proof. To prove (1), it is enough to show that all solutions to the equations

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Proof. To prove (1), it is enough to show that all solutions to the equations

$$\rho\,(\boldsymbol{v}_t + \dot{\omega}\boldsymbol{e}_3 \times \boldsymbol{x}) - \mu\Delta\boldsymbol{v} + \nabla p = \mathbf{0},\ \ \nabla\cdot\boldsymbol{v} = 0,\ \ \boldsymbol{v}(x,t)|_{\partial\mathcal{C}} = \mathbf{0}$$

$$\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2\,, \quad \dot{\boldsymbol{\gamma}} + \omega\,\boldsymbol{e}_2 = \mathbf{0}\,,$$

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Asymptotic stability of a liquid-filled pendulum (continued)

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are uniformly bounded in time. In this case the energy balance read as follows

$$\frac{1}{2}\frac{d}{dt}\left[\rho\,\|\boldsymbol{v}\|_2^2 - \mathsf{C}\,a^2 + \mathsf{C}\,(\omega - a)^2 + \beta^2\gamma_2^2\right] + \mu\|\nabla\boldsymbol{v}\|_2^2 = 0.$$

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$$\begin{aligned}
&\rho\,(\boldsymbol{v}_t + \dot{\omega}\boldsymbol{e}_3 \times \boldsymbol{x}) - \mu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{0},\ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}(x,t)|_{\partial\mathcal{C}} = \boldsymbol{0} \\
&\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2\,, \quad \dot{\boldsymbol{\gamma}} + \omega\,\boldsymbol{e}_2 = \boldsymbol{0}\,, \\
&(\boldsymbol{v}(\cdot,0), \omega(0), \boldsymbol{\gamma}(0)) \in L^2_\sigma(\mathcal{C}) \oplus \mathbb{R} \oplus \mathbb{R}^2
\end{aligned}$$

are uniformly bounded in time. In this case the energy balance read as follows

$$\frac{1}{2}\frac{d}{dt}[\overbrace{\rho\,\|\boldsymbol{v}\|_2^2 - \mathsf{C}\,a^2}^{c\|\boldsymbol{v}\|_2^2 \le \mathcal{E}_f \le \|\boldsymbol{v}\|_2^2} + \mathsf{C}\,(\omega - a)^2 + \beta^2\gamma_2^2] + \mu\|\nabla\boldsymbol{v}\|_2^2 = 0.$$

Asymptotic stability of a liquid-filled pendulum (continued)

Proof. (continued) To prove (2), we will proceed by contradiction. Assume that all solutions to

$$\begin{aligned}&\rho\left(\boldsymbol{v}_t+\dot{\omega}\boldsymbol{e}_3\times\boldsymbol{x}\right)-\mu\Delta\boldsymbol{v}+\nabla p=\boldsymbol{0},\ \nabla\cdot\boldsymbol{v}=0,\ \boldsymbol{v}(x,t)|_{\partial\mathcal{C}}=\boldsymbol{0}\\&\mathsf{C}(\dot{\omega}-\dot{a})=\beta^2\gamma_2\,,\quad\dot{\boldsymbol{\gamma}}-\omega\,\boldsymbol{e}_2=\boldsymbol{0}\,,\end{aligned}$$

satisfy $\|\boldsymbol{v}(t)\|_2^2+|\omega(t)|^2+|\boldsymbol{\gamma}(t)|^2\le M(\|\boldsymbol{v}(0)\|_2,|\omega(0)|,|\boldsymbol{\gamma}(0)|)$, for all $t\ge 0$.

Asymptotic stability of a liquid-filled pendulum (continued)

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$$\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2\,, \quad \dot{\boldsymbol{\gamma}} - \omega\,\boldsymbol{e}_2 = \mathbf{0}\,,$$

satisfy $\|\boldsymbol{v}(t)\|_2^2 + |\omega(t)|^2 + |\boldsymbol{\gamma}(t)|^2 \le M(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$, for all $t \ge 0$. From Navier-Stokes equations and $\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2$, one finds that

$$\frac{1}{2}\frac{d\mathcal{E}_f}{dt} + c_1\mathcal{E}_f \le c_2\|\boldsymbol{v}\|_2$$

Asymptotic stability of a liquid-filled pendulum (continued)

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$$\frac{1}{2}\frac{d\mathcal{E}_f}{dt} + c_1\mathcal{E}_f \leq c_2\|\boldsymbol{v}\|_2 \ \Rightarrow\ \lim_{t\to\infty}\|\boldsymbol{v}(t)\|_2 = 0.$$

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Then, the Ω-limit set $\Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$ of the dynamical system generated by $\dfrac{du}{dt} + \boldsymbol{L}\boldsymbol{u} = \mathbf{0}$ is connected, compact and invariant. Moreover, for every $(\bar{\boldsymbol{v}}, \bar{\omega}, \bar{\boldsymbol{\gamma}}) \in \Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$, one has $\bar{\boldsymbol{v}} \equiv \mathbf{0}$ and then $\dot{\bar{\omega}} = 0$, implying that $\bar{\gamma}_2 = \bar{\omega} = 0$.

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$$\frac{1}{2}[c\|\boldsymbol{v}(t)\|_2^2 + \mathsf{C}\left(\omega(t) - a(t)\right)^2 - \beta^2\gamma_2^2(t)] + \mu\int_0^t \|\nabla\boldsymbol{v}(\tau)\|_2^2\, d\tau$$
$$= \frac{1}{2}[\|\boldsymbol{v}(0)\|_2^2 + \mathsf{C}\left(\omega(t) - \bar{a}(0)\right)^2 - \beta^2\gamma_2^2(0)].$$

Proof. (continued) To prove (2), we will proceed by contradiction. Assume that all solutions to

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Then, the Ω-limit set $\Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$ of the dynamical system generated by $\dfrac{du}{dt} + \boldsymbol{L}\boldsymbol{u} = \boldsymbol{0}$ is connected, compact and invariant. Moreover, for every $(\bar{\boldsymbol{v}}, \bar\omega, \bar{\boldsymbol{\gamma}}) \in \Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$, one has $\bar{\boldsymbol{v}} \equiv \boldsymbol{0}$ and then $\dot{\bar\omega} = 0$, implying that $\bar\gamma_2 = \bar\omega = 0$. Integrating the energy balance and taking the limit as $t \to \infty$,

$$\frac{1}{2}[c\underbrace{\|\boldsymbol{v}(t)\|_2^2}_{\to 0} + \mathsf{C}\underbrace{\left(\omega(t) - a(t)\right)^2}_{\to 0} - \beta^2\underbrace{\gamma_2^2(t)}_{\to 0}] + \mu\int_0^t \|\nabla\boldsymbol{v}\|_2^2\, d\tau$$
$$= \frac{1}{2}[c\|\boldsymbol{v}(0)\|_2^2 + \mathsf{C}\left(\omega(0) - a(0)\right)^2 - \beta^2\gamma_2^2(0)].$$

Asymptotic stability of a liquid-filled pendulum (continued)

Proof. (continued) To prove (2), we will proceed by contradiction. Assume that all solutions to

$$\rho\left(\boldsymbol{v}_t + \dot{\omega}\boldsymbol{e}_3 \times \boldsymbol{x}\right) - \mu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{0},\ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}(x,t)|_{\partial\mathcal{C}} = \boldsymbol{0}$$

$$\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2\,, \quad \dot{\boldsymbol{\gamma}} - \omega\,\boldsymbol{e}_2 = \boldsymbol{0}\,,$$

satisfy $\|\boldsymbol{v}(t)\|_2^2 + |\omega(t)|^2 + |\boldsymbol{\gamma}(t)|^2 \le M(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$, for all $t \ge 0$. From Navier-Stokes equations and $\mathsf{C}(\dot{\omega} - \dot{a}) = \beta^2\gamma_2$, one finds that

$$\frac{1}{2}\frac{d\mathcal{E}_f}{dt} + c_1\mathcal{E}_f \le c_2\|\boldsymbol{v}\|_2 \;\Rightarrow\; \lim_{t\to\infty}\|\boldsymbol{v}(t)\|_2 = 0.$$

Then, the Ω-limit set $\Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$ of the dynamical system generated by $\dfrac{du}{dt} + \boldsymbol{L}\boldsymbol{u} = \boldsymbol{0}$ is connected, compact and invariant. Moreover, for every $(\bar{\boldsymbol{v}}, \bar{\omega}, \bar{\boldsymbol{\gamma}}) \in \Omega(\|\boldsymbol{v}(0)\|_2, |\omega(0)|, |\boldsymbol{\gamma}(0)|)$, one has $\bar{\boldsymbol{v}} \equiv \boldsymbol{0}$ and then $\dot{\bar{\omega}} = 0$, implying that $\bar{\gamma}_2 = \bar{\omega} = 0$. Integrating the energy balance and taking the limit as $t \to \infty$, for any initial data

$$\mu\int_0^\infty \|\nabla\boldsymbol{v}\|_2^2\,d\tau = \frac{1}{2}[c\|\boldsymbol{v}(0)\|_2^2 + \mathsf{C}\left(\omega(0) - a(0)\right)^2 - \beta^2\gamma_2^2(0)]].$$

Main Theorem on the asymptotic stability of a liquid-filled pendulum

- The steady-state solution s_0^+, representing the equilibrium configuration where the center of mass G of $\mathcal{S}$ is in its lower position, is asymptotically, exponentially stable:

  (a) There is $\rho_0 > 0$ such that if, for some $\alpha \in [3/4, 1)$,

  $$\|\boldsymbol{A}_0^\alpha \boldsymbol{v}(0)\|_2 + |\omega(0)| + |\boldsymbol{\gamma}(0)| < \rho_0 \,,$$

  then there exists a corresponding unique, global solution $(\boldsymbol{v}, \omega, \boldsymbol{\gamma})$, such that, for all $T > 0$,

  $$\boldsymbol{v} \in C((0,T]; D(\boldsymbol{A}_0)) \cap C^1((0,T]; L^2_\sigma(\mathcal{C}))\,, \quad \boldsymbol{A}_0^\alpha \boldsymbol{v} \in C([0,T]; L^2_\sigma(\mathcal{C}))\,,$$
  $$\omega \in C([0,T];\mathbb{R}) \cap C^1((0,T];\mathbb{R})\,; \quad \boldsymbol{\gamma} \in C^1([0,T];\mathbb{R}^2) \cap C^2((0,T];\mathbb{R}^2)\,;$$

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- The steady-state solution s_0^-, representing the equilibrium configuration where the center of mass G of $\mathcal{S}$ is in its higher position, is unstable.

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Remark. Note that, from (b), $\boldsymbol{\gamma}(t) \to \mathbf{0}$ as $t \to \infty$.

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In fact, from our abstract stability theorem, it follows that $\boldsymbol{\gamma}(t) \to \sigma \boldsymbol{e}_1$ as $t \to \infty$ for some $\sigma \in \mathbb{R}$.

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In fact, from our abstract stability theorem, it follows that $\boldsymbol{\gamma}(t) \to \sigma \boldsymbol{e}_1$ as $t \to \infty$ for some $\sigma \in \mathbb{R}$.
Recall that $|\boldsymbol{\gamma}(t) + \boldsymbol{e}_1| = 1$ at all times, so $\sigma^2 + 2\sigma = 0$ and $|\sigma| < \varepsilon$.

Asymptotic Behavior of the motion of a liquid-filled pendulum for LARGE INITIAL DATA

Another way of stating the stability result is to say that all solutions to

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{v} &= 0 \\ \rho\left(\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dot\omega\boldsymbol{e}_3\times\boldsymbol{x} + 2\omega\,\boldsymbol{e}_3\times\boldsymbol{v}\right) &= \mu\Delta\boldsymbol{v} - \nabla p \end{aligned}\right\} \text{ in } \mathcal{C}\times\mathbb{R}_+,$$

$$\boldsymbol{v}(x,t) = \mathbf{0} \qquad \text{on } \partial\mathcal{C}\times\mathbb{R}_+,$$

$$\mathsf{C}(\dot\omega - \dot a) = \beta^2\chi_2 \qquad \text{in } \mathbb{R}_+,$$

$$\dot{\boldsymbol{\chi}} + \omega\,\boldsymbol{e}_3\times\boldsymbol{\chi} = \mathbf{0} \qquad \text{in } \mathbb{R}_+,$$

corresponding to “sufficiently smooth” initial data that are “sufficiently close” to the equilibrium configuration s_0^+ must remain “close” to s_0^+ and eventually converge to it at an exponential rate.

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corresponding to “sufficiently smooth” initial data that are “sufficiently close” to the equilibrium configuration s_0^+ must remain “close” to s_0^+ and eventually converge to it at an exponential rate.

- The same conclusion holds in the more general class of *weak* solutions for data that not only are less regular, but also not necessarily “close” to the stable equilibrium configuration s_0^+.

The class of weak solutions

Definition. The triple $(\boldsymbol{v}, \omega, \boldsymbol{\chi})$ is a *weak solution* if it meets the following requirements:

(a) $\boldsymbol{v} \in C_w([0,\infty); L^2_\sigma(\mathcal{C})) \cap L^\infty(0,\infty; L^2_\sigma(\mathcal{C})) \cap L^2(0,\infty; W_0^{1,2}(\mathcal{C}))$;

(b) $\omega \in C^0([0,\infty)) \cap L^\infty(0,\infty)$, $\boldsymbol{\chi} \in C^1([0,\infty); \mathsf{S}^1)$;

(c) Strong Energy Inequality: for all $t \geq s$ and a.a. $s \geq 0$ including $s = 0$,

$$\mathcal{E}(t) + \mathcal{U}(t) + \mu \int_t^s \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau \leq \mathcal{E}(s) + \mathcal{U}(s)$$

where

$$\mathcal{E} := \left[\rho \, \|\boldsymbol{v}\|_2^2 - \mathsf{C} \, a^2 + \mathsf{C} \, (\omega - a)^2\right] \quad \text{(kinetic energy)}$$

and

$$\mathcal{U} := -\beta^2 \chi_1 \quad \text{(potential energy).}$$

(d) $(\boldsymbol{v}, \omega, \boldsymbol{\chi})$ satisfies the equations of motion in the sense of distributions and the boundary conditions in the trace sense.

Preliminary results (G. P. Galdi & G.M. (2016))

For any given initial data $(\boldsymbol{v}_0, \omega_0, \boldsymbol{\chi}_0) \in L^2_\sigma(\mathcal{C}) \times \mathbb{R} \times \mathsf{S}^1$, there exists at least one corresponding weak solution $(\boldsymbol{v}, \omega, \boldsymbol{\chi})$ and it satisfies $\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_2 = 0$.

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The previous result guarantees that, as time approaches to infinity, the liquid will reach a state of motion which is the rest relative to $\mathcal{B}$. Thus, the system will move as a whole rigid body. However, at this stage, we do not know whether the ultimate motion of the whole system will be a steady-state (i.e. the rest) or a time-dependent motion.

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There exists a time t_0 (depending on the solution) such that, setting $I_{t_0,T} = (t_0, t_0 + T)$,

$$\boldsymbol{v} \in C^0(\overline{I_{t_0,T}}; W^{1,2}_0(\mathcal{C})) \cap L^\infty(t_0, \infty; W^{1,2}_0(\mathcal{C})) \cap L^2(I_{t_0,T}; W^{2,2}(\mathcal{C})),$$
$$\boldsymbol{v}_t \in L^2(I_{t_0,T}; H(\mathcal{C})), \quad \omega \in W^{1,\infty}(I_{t_0,T}), \quad \boldsymbol{\chi} \in W^{2,\infty}(I_{t_0,T}; \mathsf{S}^1),$$

for all $T > 0$. Moreover, there is $p \in L^2(I_{t_0,T}; W^{1,2}(\mathcal{C}))$, all $T > 0$, such that $(\boldsymbol{v}, p, \omega, \boldsymbol{\chi})$ satisfies the equations of motion a.e. in $\mathcal{C} \times (t_0, \infty)$.

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In addition: $\lim\limits_{t\to\infty} (\|\boldsymbol{v}(t)\|_{2,2} + \|\boldsymbol{v}_t(t)\|_2 + |\omega(t)|) = 0\,.$

For all initial data such that $\rho\,\|\boldsymbol{v}_0\|_2^2 + \mathsf{C}\,(\omega_0 - a(0))^2 < 2\beta^2\,(1 + \chi_{1,0})$, we have also $\lim_{t\to\infty} |\boldsymbol{\chi}(t) - \boldsymbol{e}_1| = 0$.

Rate of decay to equilibrium (G. P. Galdi & G. M. (2017))

Theorem

Let the initial data satisfy the condition

$$\rho\,\|\boldsymbol{v}_0\|_2^2 + \mathsf{C}\,(\omega_0 - a(0))^2 < 2\beta^2\,(1+\chi_{1,0}).$$

Then, for any corresponding weak solution $(\boldsymbol{v},\omega,\boldsymbol{\chi})$, there are t_0, C_1, possibly depending on the solution, and $C_2 > 0$ such that

$$\|\boldsymbol{v}(t)\|_{2,2} + \|\boldsymbol{v}_t(t)\|_2 + |\omega(t)| + |\dot{\omega}(t)| + |\boldsymbol{\chi}(t) - \boldsymbol{e}_1| \le C_1\,e^{-C_2\,t}\,, \quad \textit{for all } t \ge t_0\,.$$

THANK YOU!