The Semiclassical Sine-Gordon Equation and Rational Solutions of Painlevé-II

Peter D. Miller

Department of Mathematics
University of Michigan

Joint work with Robert Buckingham (Cincinnati). Supported by the National Science Foundation.

Midwestern Workshop on Asymptotic Analysis
Indiana University, Bloomington IN
October 10, 2015

The Sine-Gordon Equation

Semiclassical limit for pure-impulse initial data.

Consider the following Cauchy problem for $u^\epsilon = u^\epsilon(x,t)$:

$$\epsilon^2 u_{tt}^\epsilon - \epsilon^2 u_{xx}^\epsilon + \sin(u^\epsilon) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u^\epsilon(x,0) = F(x) = 0, \qquad \epsilon u_t^\epsilon(x,0) = G(x).$$

Here $\epsilon > 0$ is a small parameter, and $G \in \mathscr{S}(\mathbb{R})$ is independent of ϵ. Setting $t = \epsilon T$, the equation looks like a perturbed simple pendulum:

$$u_{TT}^\epsilon + \sin(u^\epsilon) = \epsilon^2 u_{xx}^\epsilon.$$

The unperturbed problem conserves the total energy

$$E = \frac{1}{2}(u_T^\epsilon)^2 + (1 - \cos(u^\epsilon)).$$

Expectation: for $T = \mathcal{O}(1)$, the pendulum at x undergoes approximate

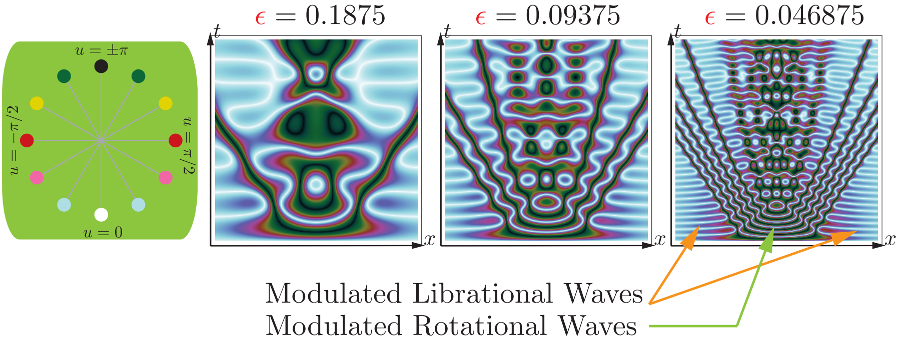
- *librational motion* ($|u^\epsilon| < \pi$) if $E = E(x) < 2$
- *rotational motion* (u^ϵ grows without bound) if $E = E(x) > 2$.

The Sine-Gordon Equation

R. Buckingham and M., *Mem. AMS* **225**, 1–152, 2013.

When $t = 0$, the energy is $E = \frac{1}{2}G(x)^2$. One can prove rigorously that a sufficiently strong impulse profile G produces both types of motion, with modulations subject to Whitham's theory, separated by critical points $x = x_{\text{crit}}$ where $G(x_{\text{crit}}) = \pm 2$.

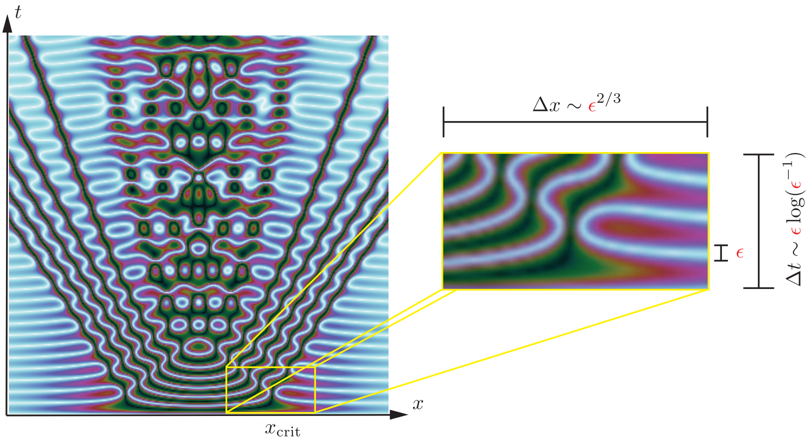
$G(x) = -3\,\text{sech}(x)$ $\qquad$ $(x,t) \in (-2.5, 2.5) \times (0, 5)$



The Sine-Gordon Equation

R. Buckingham and M., *J. Anal. Math.* **118**, 397–492, 2012.

Consider the behavior of u^{ϵ} near a critical point $x = x_{\text{crit}}$:



Let $\nu := [12G'(x_{\text{crit}})]^{-1} > 0$ and set $\Delta x := x - x_{\text{crit}}$.

The Sine-Gordon Equation

R. Buckingham and M., *J. Anal. Math.* **118**, 397–492, 2012.

Describing the asymptotics near the critical point $x = x_{\text{crit}}$ and $t = 0$ requires introducing a certain family of rational functions.

Set $\mathcal{U}_0(y) := 1$ and $\mathcal{V}_0(y) := -y/6$. Generate $\{\mathcal{U}_m, \mathcal{V}_m\}_{m\in\mathbb{Z}}$ by the forward recursion

$$\mathcal{U}_{m+1}(y) := -\frac{1}{6}y\mathcal{U}_m(y) - \frac{\mathcal{U}_m'(y)^2}{\mathcal{U}_m(y)} + \frac{1}{2}\mathcal{U}_m''(y) \quad \text{and} \quad \mathcal{V}_{m+1}(y) := \frac{1}{\mathcal{U}_m(y)}$$

and the backward recursion

$$\mathcal{U}_{m-1}(y) := \frac{1}{\mathcal{V}_m(y)} \quad \text{and} \quad \mathcal{V}_{m-1}(y) := \frac{1}{2}\mathcal{V}_m''(y) - \frac{\mathcal{V}_m'(y)^2}{\mathcal{V}_m(y)} - \frac{1}{6}y\mathcal{V}_m(y).$$

The Sine-Gordon Equation

R. Buckingham and M., *J. Anal. Math.* **118**, 397–492, 2012.

Theorem (Sine-Gordon Behavior Near Pendulum Separatrix)

Fix an integer m and assume that (x,t) lies in the horizontal strip S_m in the (x,t)-plane given by the inequality

$$\left| t - \frac{2}{3} m\epsilon \log(\epsilon^{-1}) \right| \leq \frac{1}{3}\epsilon \log(\epsilon^{-1}).$$

Suppose also that $\Delta x = \mathcal{O}(\epsilon^{2/3})$. Then as $\epsilon \to 0$,

$$\cos(\tfrac{1}{2}u^{\epsilon}(x,t)) = (-1)^m \operatorname{sgn}(\mathcal{U}_m(y)) \operatorname{sech}(T) + o(1)$$
$$\sin(\tfrac{1}{2}u^{\epsilon}(x,t)) = (-1)^{m+1} \tanh(T) + o(1), \quad \textit{where}$$

$$T := \frac{t}{\epsilon} - 2m \log\left(\frac{4\nu^{1/3}}{\epsilon^{1/3}}\right) + \log|\mathcal{U}_m(y)|, \quad \textit{and} \quad y := \frac{\Delta x}{2\nu^{1/3}\epsilon^{2/3}}.$$

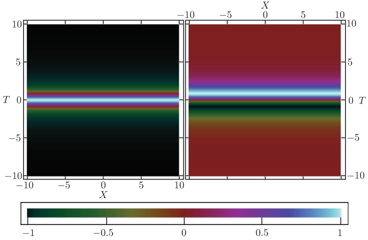
The Sine-Gordon Equation

R. Buckingham and M., *J. Anal. Math.* **118**, 397–492, 2012.

The leading terms determine a limiting X-independent exact solution u of the unscaled equation

$$u_{TT} - u_{XX} + \sin(u) = 0.$$

This exact solution represents a superluminal (infinite velocity) kink with unit magnitude *topological charge* $\sigma := \mathrm{sgn}(\mathcal{U}_m(y))$.

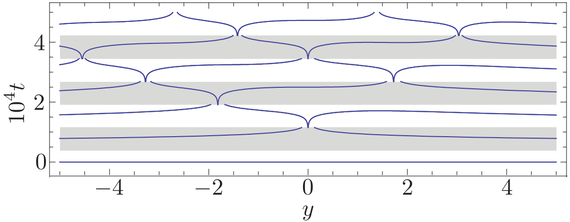


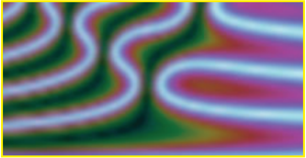
$\cos(u(T))$ $\sin(u(T))$

The Sine-Gordon Equation

R. Buckingham and M., *J. Anal Math.* **118**, 397–492, 2012

But T includes weak dependence on the spatial variable y, and hence each kink is slowly modulated in the direction parallel to the wavefront. The center ($T = 0$) is a vertical translate of the graph of $-\log|\mathcal{U}_m(y)|$:





The strips $S_0, \ldots, S_6$ in the (y, t)-plane for $\epsilon = 10^{-5}$ and $4\nu^{1/3} = 1$, and the curve $T = 0$ in each strip.

Since $t = \mathcal{O}(1)$ means $m \to \infty$, knowledge of the large-m behavior of the rational functions $\mathcal{U}_m(y)$ will be useful to determine how the kink pattern matches onto larger time dynamics.

Rational Painlevé-II Functions

It may seem miraculous given what has been explained so far, but. . .

It turns out that $(\mathcal{U}, \mathcal{V}) = (\mathcal{U}_m, \mathcal{V}_m)$ satisfy for each m the coupled system of second-order Painlevé II-type equations

$$\mathcal{U}''(y)+2\mathcal{U}(y)^2\mathcal{V}(y)+\frac{1}{3}y\mathcal{U}(y) = 0 \quad \text{and} \quad \mathcal{V}''(y)+2\mathcal{U}(y)\mathcal{V}(y)^2+\frac{1}{3}y\mathcal{V}(y) = 0.$$

Moreover, the logarithmic derivative

$$\mathcal{P}(y) = \mathcal{P}_m(y) := \frac{\mathcal{U}_m'(y)}{\mathcal{U}_m(y)}$$

is a rational solution of the inhomogeneous Painlevé-II equation (PII$_\alpha$)

$$\mathcal{P}''(y) = 2\mathcal{P}(y)^3 + \frac{2}{3}y\mathcal{P}(y) + \frac{2}{3}\alpha, \quad \alpha \in \mathbb{C}$$

in the special case that $\alpha = -m \in \mathbb{Z}$.

Rational Painlevé-II Functions

History and applications.

The functions $\mathcal{P}_m$ were

- discovered as solutions of PII$_{-m}$ by Yablonskii (1959) and Vorob'ev (1965), and
- studied via Bäcklund transformations by Airault (1979).

Airault showed that the condition $\alpha \in \mathbb{Z}$ is *necessary* for PII$_\alpha$ to have a rational solution, and Murata (1985) has shown that $\mathcal{P}_m$ is the *unique* rational solution of PII$_{-m}$.

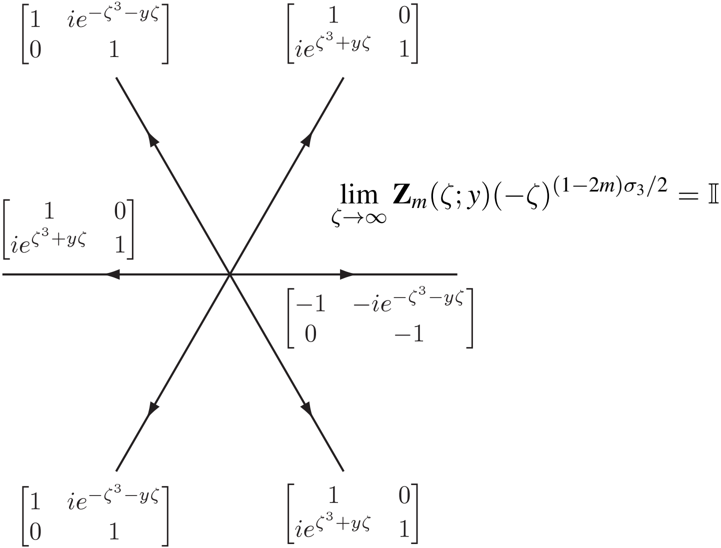
In addition to sine-Gordon, the rational Painlevé-II functions arise in

- the theory of *steady electrolysis* (Bass 1964 and Rogers, Bassom, & Schief 1999),
- *string theory* (Johnson 2006), and
- the theory of *plane equilibrium fluid vortex configurations* (Clarkson 2009).

Rational Painlevé-II Functions

Connection with semiclassical sine-Gordon. Parametrix Riemann-Hilbert problem.

The IST for sG near criticality boils down to this problem for $\mathbf{Z}_m(\zeta;y)$:



The quantities of interest in sine-Gordon theory are obtained from $\mathbf{Z}_m(\zeta;y)$ by expansion for large ζ:

$$\mathcal{U}_m(y) = A_{m,12}(y) \quad \text{and} \quad \mathcal{P}_m(y) = A_{m,22}(y) - \frac{B_{m,12}(y)}{A_{m,12}(y)}$$

where the matrices $\mathbf{A}_m(y)$ and $\mathbf{B}_m(y)$ are obtained from the expansion:

$$\mathbf{Z}_m(\zeta;y)(-\zeta)^{(1-2m)\sigma_3/2} = \mathbb{I} + \mathbf{A}_m(y)\zeta^{-1} + \mathbf{B}_m(y)\zeta^{-2} + \mathcal{O}(\zeta^{-3}), \quad \zeta \to \infty.$$

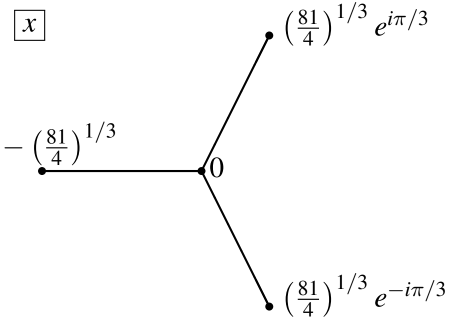
One can prove directly from the conditions governing $\mathbf{Z}_m(\zeta;y)$ that $\mathcal{U}_m$ and $\mathcal{P}_m$ are rational functions satisfying the desired recursion and differential equations. (*This explains the aforementioned miracles.*)

Now turn the problem around: consider using Deift-Zhou steepest descent analysis of $\mathbf{Z}_m(\zeta;y)$ in the limit of large m to deduce large-degree asymptotics of $\mathcal{U}_m$ and $\mathcal{P}_m$.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

The simplest result to state involves an analytic function $S = S(x)$ defined as follows: $S(x)$ is the unique solution of the cubic equation $3S^3 + 4xS + 8 = 0$ that is analytic for $x \in \mathbb{C} \setminus \Sigma_S$ where Σ_S is the contour



Note that $S(x) = -2x^{-1} + \mathcal{O}(x^{-4})$ as $x \to \infty$ and $S(x)$ is real for $x \in \mathbb{R}$.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

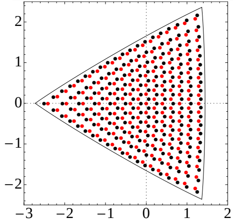
Theorem (Exterior Asymptotics)

There exists a piecewise-analytic simple closed curve ∂T such that uniformly for $x = y/(m-\frac{1}{2})^{2/3}$ bounded outside ∂T (and also for x as close as $\log(m)/m$ from an edge — but not a corner), as $m \to +\infty$,

$$m^{-2m/3}e^{-m\lambda(x)}\mathcal{U}_m = \dot{\mathcal{U}}(x) + \mathcal{O}(m^{-1}), \quad \dot{\mathcal{U}}(x) := e^{xS(x)/6},$$

$$m^{-1/3}\mathcal{P}_m = \dot{\mathcal{P}}(x) + \mathcal{O}(m^{-1}), \quad \dot{\mathcal{P}}(x) := -\frac{1}{2}S(x),$$

where the normalizing exponent for $\mathcal{U}$ is $\lambda(x) := \frac{1}{4}S(x)^3 - \log(3S(x))$.



Poles and zeros of $\mathcal{U}_m(y)$ in the x-plane for $m = 20$ and the curve ∂T. The opening angle of ∂T at each corner is exactly $2\pi/5$. ∂T is (part of) the zero locus of a harmonic function explicit in S.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

Boutroux ansatz method: Let x_0 be a fixed complex number and set

$$y = (m - \tfrac{1}{2})^{2/3}(x_0 + (m - \tfrac{1}{2})^{-1}w).$$

Writing $\mathcal{P}_m(y) = (m - \frac{1}{2})^{1/3}q(w)$ converts the exact equation

$$\mathcal{P}_m''(y) = 2\mathcal{P}_m(y)^3 + \frac{2}{3}y\mathcal{P}_m(y) - \frac{2}{3}m$$

into the form

$$q''(w) = 2q(w)^3 + \frac{2}{3}x_0q(w) - \frac{2}{3} + (m - \tfrac{1}{2})^{-1}\left[\frac{2}{3}wq(w) - \frac{1}{3}\right].$$

Neglecting the formally small red terms $\implies$ $q(w)$ is an elliptic function with modulus depending on x_0.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

The formal Boutroux approximation is valid as long as x_0 lies within the interior of T, the "elliptic region".

Theorem (Interior Asymptotics)

There exists a smooth but non-analytic function $\Lambda : T \to \mathbb{C}$ such that with $y = (m - \frac{1}{2})^{2/3}x$ and $x = x_0 + (m - \frac{1}{2})^{-1}w$, as $m \to +\infty$,

$$m^{-2m/3}e^{-m\Lambda(x)}\mathcal{U}_m = \frac{\dot{\mathcal{U}}_m(w; x_0)}{1 + \mathcal{O}(m^{-1}\dot{\mathcal{U}}_m(w; x_0))},$$

$$m^{-1/3}\mathcal{P}_m = \frac{\dot{\mathcal{P}}_m(w; x_0)}{1 + \mathcal{O}(m^{-1}\dot{\mathcal{P}}_m(w; x_0))},$$

both hold uniformly for x_0 in compact subsets of T and w bounded, where $\dot{\mathcal{U}}_m(w; x_0)$ and $\dot{\mathcal{P}}_m(w; x_0)$ are explicitly constructed in terms of the Riemann theta function of a uniquely determined elliptic curve $\Gamma(x_0)$.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

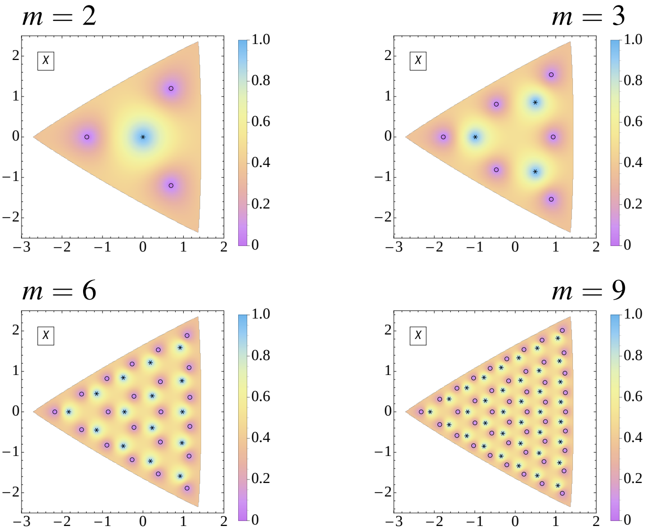
R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

Some notes:

- For each $x_0 \in T$, $\dot{\mathcal{P}}_m(w; x_0)$ is an elliptic function of w that solves the Boutroux ansatz differential equation.
- Accuracy even near poles is obtained using Bäcklund transformations.
- Pole/zero locations accurate to $\mathcal{O}(m^{-2})$ in x; spacing scales as m^{-1}.
- Interpretation of two-variable approximations:
  - x_0 is a coordinate on the base manifold T. Setting $w = 0$ gives a *uniform approximation* that is not meromorphic in $x_0 = x$.
  - w is a coordinate on the tangent space to T at x_0. Fixing x_0 and varying w gives a *tangent approximation* that is meromorphic in w but only locally accurate.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.



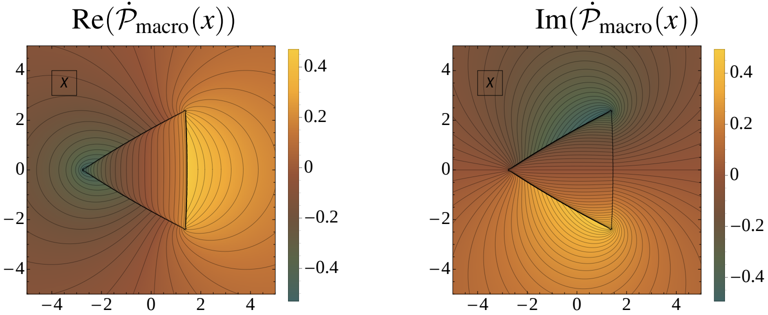
Plots of $\frac{2}{\pi}\arctan(|\dot{\mathcal{U}}_m(0;x)|)$ *with zeros (○) and poles (∗) of* $\mathcal{U}_m$.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

From our approximate formulae:

- We prove that the function $m^{-1/3}\mathcal{P}_m((m-\frac{1}{2})^{2/3}x)$ converges to a continuous limit $\dot{\mathcal{P}}_{\mathrm{macro}}(x)$ in the distributional topology of $\mathscr{D}'(\mathbb{C}\setminus\partial T)$, as well as in the (suitably PV-regularized) distributional topology of $\mathscr{D}'(\mathbb{R}\setminus\{x_c,x_e\})$, where $(x_c,x_e)=T\cap\mathbb{R}$.



Note $\overline{\partial}\dot{\mathcal{P}}_{\mathrm{macro}}(x)\neq 0$ for $x\in T$.

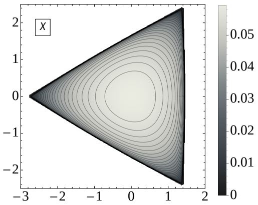
$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.

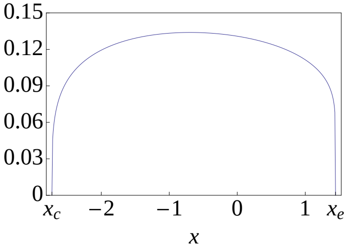
From our approximate formulae:

- We calculate the asymptotic planar density (at $x \in T$) and linear density (at $x \in T \cap \mathbb{R}$) of poles of $\mathcal{U}_m$. Taking out a factor of m^2, these are:

Planar Density $\sigma_\mathrm{P}(x)$



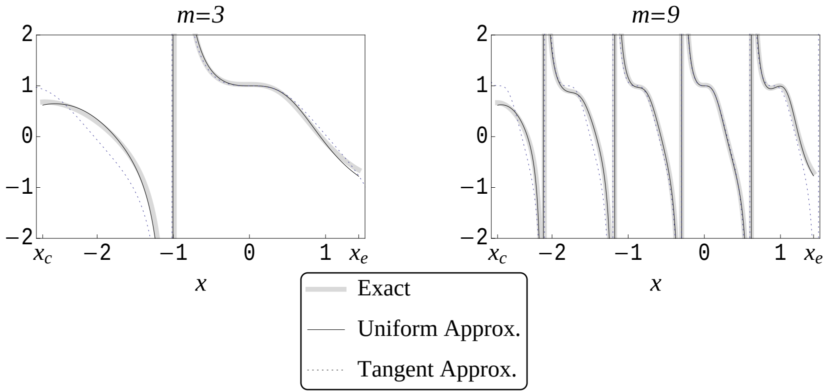
Linear Density $\sigma_\mathrm{L}(x)$



$\sigma_\mathrm{P}(x)$ is inversely proportional to the real area of the Jacobian of $\Gamma(x)$. $\sigma_\mathrm{L}(x)$ is inversely proportional to the real period of the elliptic function $\dot{\mathcal{P}}_m$ as a function of w.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

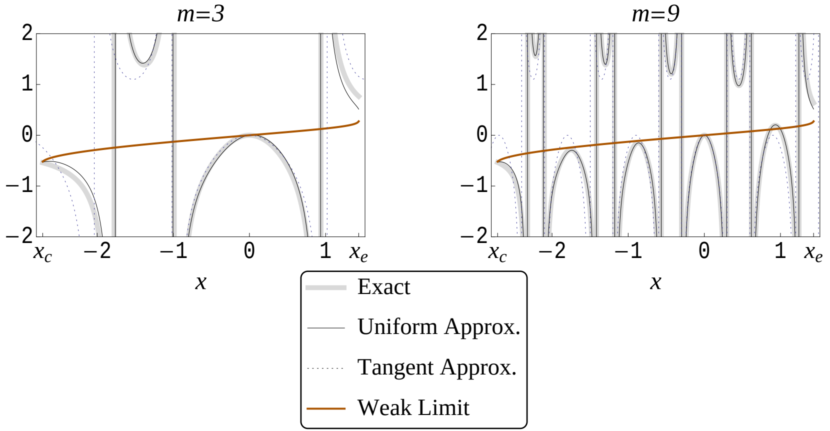
R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.



Quantitative comparison for $x \in T \cap \mathbb{R}$ *of* $m^{-2m/3}e^{-m\Lambda(x)}\mathcal{U}_m((m-\frac{1}{2})^{2/3}x)$*, the uniform approximation* $\dot{\mathcal{U}}_m(0;x)$*, and the tangent approximation based at the origin* $\dot{\mathcal{U}}_m((m-\frac{1}{2})x;0)$*.*

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Noncritical Analysis

R. Buckingham and M., *Nonlinearity* **27**, 2489–2578, 2014.



Quantitative comparison for $x \in T \cap \mathbb{R}$ *of* $m^{-1/3}\mathcal{P}_m((m-\frac{1}{2})^{2/3}x)$*, the uniform approximation* $\dot{\mathcal{P}}_m(0;x)$*, the tangent approximation based at the origin* $\dot{\mathcal{P}}_m((m-\frac{1}{2})x;0)$*, and the weak limit* $\dot{\mathcal{P}}_{\mathrm{macro}}(x)$*.*

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Critical Analysis

R. Buckingham and M., *Nonlinearity* **28**, 1539–1596, 2015.

What does "critical behavior" of the rational Painlevé-II functions mean?

- The preceding analysis of the rational Painlevé-II functions fails for $x \in \partial T$, and it fails to be uniform for x sufficiently close to ∂T.
- When x approaches ∂T at some rate while $m \to \infty$, new critical phenomena occur, requiring the modification of the steepest descent analysis of $\mathbf{Z}_m(\zeta; y)$ via the installation of specialized local parametrices.

This is in complete analogy with how the rational Painlevé-II functions arose from the IST Riemann-Hilbert problem for sine-Gordon in the first place.

The critical behavior of $\mathcal{U}_m$ and $\mathcal{P}_m$ is different depending on whether x is close to a smooth point of an edge of ∂T, or whether x is close to a corner point of ∂T.

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Critical Analysis

R. Buckingham and M., *Nonlinearity* **28**, 1539–1596, 2015.

While our work covers both edges and corners, we give here the result for x near a corner point of ∂T. By a rotational symmetry, it is sufficient to consider the negative real corner point $x = x_c = -(9/2)^{2/3}$.
Our results are formulated in terms of the famous (real) *tritronquée* solution $Y(t)$ of the Painlevé-I equation:

$$Y''(t) = 6Y(t)^2 + t$$

that is uniquely specified by the asymptotic behavior

$$Y(t) = -\left(-\frac{t}{6}\right)^{1/2} + \mathcal{O}(t^{-2}), \quad t \to \infty, \quad |\arg(-t)| \leq \frac{4}{5}\pi - \delta, \quad \delta > 0.$$

Dubrovin, Grava, and Klein conjectured (2009), and Costin, Huang, and Tanveer proved (2014), that $Y(t)$ is analytic for $|\arg(-t)| < 4\pi/5$ *without the condition* $t \to \infty$. Associated with Y is its *Hamiltonian*

$$H(t) := \frac{1}{2}Y'(t)^2 - 2Y(t)^3 - tY(t).$$

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Critical Analysis

R. Buckingham and M., *Nonlinearity* **28**, 1539–1596, 2015.

Theorem (Corner Asymptotics)

Let $\mathcal{K}$ be a compact set in the t-plane containing no poles of $Y(\cdot)$. Then

$$\left(\frac{m}{6}\right)^{-2m/3} e^{1/2-m/3} e^{m(x-x_c)/6^{1/3}} \mathcal{U}_m\left((m-\tfrac{1}{2})^{2/3}x\right) = 1 + \frac{2^{6/15}}{m^{1/5}}H(t) + \mathcal{O}\left(\frac{1}{m^{2/5}}\right)$$

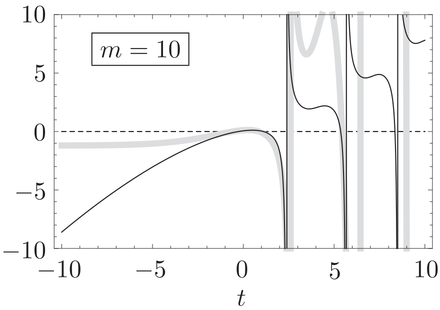
$$m^{-1/3}\mathcal{P}_m\left((m-\tfrac{1}{2})^{2/3}x\right) = -\frac{1}{6^{1/3}} - \frac{1}{m^{2/5}}\frac{2^{7/15}}{3^{1/3}}Y(t) + \mathcal{O}\left(\frac{1}{m^{3/5}}\right)$$

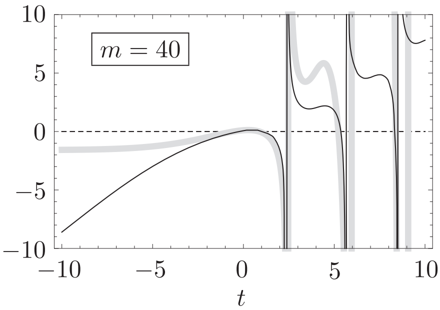
both hold in the limit $m \to +\infty$, uniformly for $t \in \mathcal{K}$, where

$$t := \frac{2^{1/15}}{3^{1/3}} m^{4/5}(x - x_c).$$

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Critical Analysis

R. Buckingham and M., *Nonlinearity* **28**, 1539–1596, 2015.

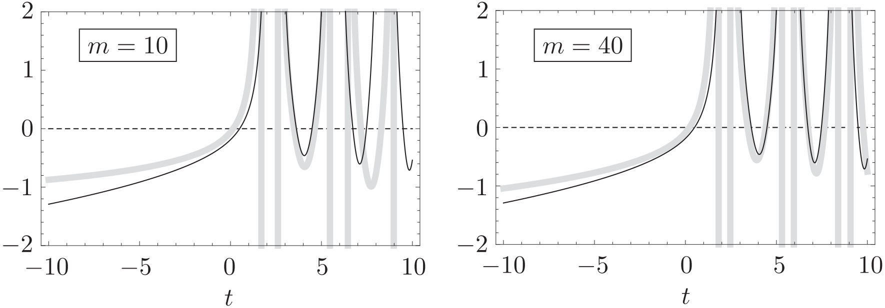




The function $2^{-6/15}m^{1/5}((m/6)^{-2m/3}e^{1/2-m/3}e^{m(x-x_c)/6^{1/3}}\mathcal{U}_m - 1)$ *(thick gray curves) and the tritronquée Hamiltonian* H *(thin black curves) both plotted as functions of* t*. (*H *was computed using the "pole field solver" of Fornberg and Weideman (2011).)*

$\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m: Critical Analysis

R. Buckingham and M., *Nonlinearity* **28**, 1539–1596, 2015.



The function $-2^{-7/15}3^{1/3}m^{2/5}(m^{-1/3}\mathcal{P}_m + 6^{-1/3})$ *(thick gray curves) and the tritronquée solution* Y *(thin black curves) both plotted as functions of* t*. (*Y *was computed using the "pole field solver" of Fornberg and Weideman (2011).)*

Conclusion

- In a semiclassical multi-scaling limit solutions to the sine-Gordon equation with initial data crossing the pendulum separatrix exhibit a universal structure near the crossing points. Superluminal kinks are centered along the real graphs of the rational functions $\mathcal{U}_m$ associated with the Painlevé-II-α equation.
- The rational Painlevé-II functions also show up in diverse physical applications including electrolysis, string theory, and the interaction of fluid vortices.
- The common link between sine-Gordon and Painlevé-II is a parametrix Riemann-Hilbert problem for $\mathbf{Z}_m(\zeta; y)$ that admits detailed asymptotic analysis in the limit $m \to \infty$, yielding useful and interesting asymptotic formulae for the rational Painlevé-II functions.
- Some of our noncritical results were obtained more recently in a different way by Bertola and Bothner (`arxiv:1401.1408`).

Thank You!

Conclusion

- In a semiclassical multi-scaling limit solutions to the sine-Gordon equation with initial data crossing the pendulum separatrix exhibit a universal structure near the crossing points. Superluminal kinks are centered along the real graphs of the rational functions $\mathcal{U}_m$ associated with the Painlevé-II-α equation.
- The rational Painlevé-II functions also show up in diverse physical applications including electrolysis, string theory, and the interaction of fluid vortices.
- The common link between sine-Gordon and Painlevé-II is a parametrix Riemann-Hilbert problem for $\mathbf{Z}_m(\zeta; y)$ that admits detailed asymptotic analysis in the limit $m \to \infty$, yielding useful and interesting asymptotic formulae for the rational Painlevé-II functions.
- Some of our noncritical results were obtained more recently in a different way by Bertola and Bothner (`arxiv:1401.1408`).

Thank You!