# Method of series expansions for orthogonal polynomials

Erwin Miña-Díaz

University of Mississippi

(Joint work with James Henegan, University of Mississippi)

If $\mu$ is a finite, positive measure, whose support is a compact subset of the complex plane containing infinitely many points, the associated orthonormal polynomials are uniquely determined by

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n \geq 0,$$

$$\int p_n(z)\overline{p_m(t)}d\mu(z) = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

If $\mu$ is a finite, positive measure, whose support is a compact subset of the complex plane containing infinitely many points, the associated orthonormal polynomials are uniquely determined by

$$p_n(z) := \kappa_n z^n + \cdots\ , \quad \kappa_n > 0, \quad n \geq 0,$$

$$\int p_n(z)\overline{p_m(t)}d\mu(z) = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

The corresponding monic orthogonal polynomials are

$$P_n(z) = \kappa_n^{-1} p_n(z), \quad n \geq 0.$$

The aim of this talk is to communicate an idea that, **in some instances**, can be used to extricate the asymptotic behavior of $\{p_n(z)\}$ out of its reproducing kernel

$$K(z, \zeta) = \sum_{n=0}^{\infty} p_n(z)\overline{p_n(\zeta)}$$

The reproducing property is that

$$Q(z) = \int K(z, \zeta)Q(\zeta)d\mu(\zeta)$$

for every polynomial $Q$.

The method has its own limitations, but I believe some ideas will be fruitful in contexts other than those discussed here.

How can we find the kernel

$$K(z, \zeta) = \sum_{n=0}^{\infty} p_n(z)\overline{p_n(\zeta)}$$

if we don't know the polynomials $P_n$?

How can we find the kernel

$$K(z, \zeta) = \sum_{n=0}^{\infty} p_n(z)\overline{p_n(\zeta)}$$

if we don't know the polynomials $P_n$?

Sometimes, the polynomials are a dense subset of some Hilbert space of analytic functions in which point evaluation functionals $f \mapsto f(\zeta)$ are bounded and the unique function $K(z, \zeta)$ such that

$$f(\zeta) =< f, K(\cdot, \zeta) >$$

happens to be

$$K(z, \zeta) = \sum_{n=0}^{\infty} p_n(z)\overline{p_n(\zeta)}.$$

Here are some examples.

- If $d\mu = \frac{1}{2\pi}|dz|$ is the normalized arclength measure on the unit circle $|z| = 1$, then

$$K(z, \zeta) = \frac{1}{1 - z\overline{\zeta}}, \quad |z|, |\zeta| < 1$$

Here are some examples.

- If $d\mu = \frac{1}{2\pi}|dz|$ is the normalized arclength measure on the unit circle $|z| = 1$, then
$$K(z, \zeta) = \frac{1}{1 - z\bar{\zeta}}, \quad |z|, |\zeta| < 1$$
- If $d\mu = \frac{1}{\pi} dA$ is the normalized area measure on the unit disk $|z| < 1$, then
$$K(z, \zeta) = \frac{1}{[1 - z\bar{\zeta}]^2}, \quad |z|, |\zeta| < 1$$

Here are some examples.

- If $d\mu = \frac{1}{2\pi}|dz|$ is the normalized arclength measure on the unit circle $|z| = 1$, then
$$K(z,\zeta) = \frac{1}{1 - z\overline{\zeta}}, \quad |z|, |\zeta| < 1$$
- If $d\mu = \frac{1}{\pi} dA$ is the normalized area measure on the unit disk $|z| < 1$, then
$$K(z,\zeta) = \frac{1}{[1 - z\overline{\zeta}]^2}, \quad |z|, |\zeta| < 1$$
- If $d\mu = \frac{1}{2\pi}|D(z)|^2|dz|$ on $|z| = 1$, where $D(z)$ is analytic and never zero on $|z| \leq 1$, then
$$K(z,\zeta) = \frac{1}{[1 - z\overline{\zeta}]D(z)\overline{D(\zeta)}}, \quad |z|, |\zeta| < 1$$

Here are some examples.

- If $d\mu = \frac{1}{2\pi}|dz|$ is the normalized arclength measure on the unit circle $|z| = 1$, then
$$K(z,\zeta) = \frac{1}{1 - z\overline{\zeta}}, \quad |z|, |\zeta| < 1$$
- If $d\mu = \frac{1}{\pi}dA$ is the normalized area measure on the unit disk $|z| < 1$, then
$$K(z,\zeta) = \frac{1}{[1 - z\overline{\zeta}]^2}, \quad |z|, |\zeta| < 1$$
- If $d\mu = \frac{1}{\pi}|D(z)|^2 dA(z)$ on $|z| \leq 1$, where $D(z)$ is analytic and never zero on $|z| < 1$, then
$$K(z,\zeta) = \frac{1}{[1 - z\overline{\zeta}]^2 D(z)\overline{D(\zeta)}}, \quad |z|, |\zeta| < 1$$

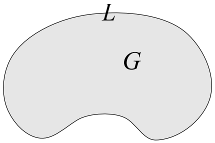
Here are some examples.

- If $d\mu = \frac{1}{\pi}|D(z)|^2|\prod_{k=1}^m (z-a_k)|^2|dz|$ on $|z|=1$, with $|a_k|<1$, then

$$K(z,\zeta) = \frac{1}{[1-z\overline{\zeta}]^2 D(z)\overline{D(\zeta)}} \frac{1}{\left[\prod_{k=1}^m (1-\overline{a_k}z)(1-a_k\overline{\zeta})\right]}$$

Here are some examples.

- If $d\mu = \frac{1}{\pi} dA$ is the area measure restricted to the interior $G$ of some Jordan curve $L$, then




$$K(z, \zeta) = \frac{\varphi(z)\overline{\varphi(\zeta)}}{[1 - \varphi(z)\overline{\varphi(\zeta)}]^2}, \quad z, \zeta \in G,$$

where $\varphi$ is any conformal map of $G$ onto the unit disk $|z| < 1$.

We will illustrate the "blue print" to follow by considering the case of polynomials orthogonal over circular multiply connected domains.

Two things worth highlighting are that orthogonality is with respect to planar measure and the domain of orthogonality is multiply connected. There are not many resources to handle these features.

We will illustrate the “blue print” to follow by considering the case of polynomials orthogonal over circular multiply connected domains.

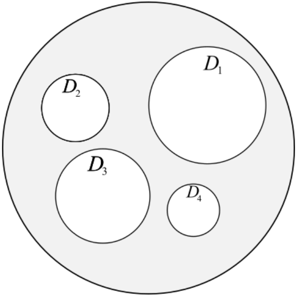
Two things worth highlighting are that orthogonality is with respect to planar measure and the domain of orthogonality is multiply connected. There are not many resources to handle these features.

Some notation we will use:

$$\mathbb{D}_r := \{z : |z| < r\}, \qquad \mathbb{T}_r := \{z : |z| = r\}$$

# Circular Multiply Connected Domains (CMCDs)

Let $\mathbb{D}_1$ be the unit disk, and let $D_1, D_2, \ldots, D_N$ be $N \geq 1$ closed disks inside the unit disk that lie exterior to each other:




Consider the multiply connected domain

$$\mathcal{D} := \mathbb{D}_1 \setminus \left( \cup_{k=1}^{N} D_k \right).$$

We refer to $\mathcal{D}$ as a *circular multiply connected domain*, or a CMCD.

# Circular Multiply Connected Domains (CMCDs)

Then, we consider polynomials $p_n(z)$, $n = 0, 1, 2, \ldots,$ orthonormal over $\mathcal{D}$:

$$p_n(z) := \kappa_n z^n + \cdots, \quad n \geq 0,$$

$$\frac{1}{\pi} \int_{\mathcal{D}} p_n(z) \overline{p_m(t)} dA(z) = \begin{cases} 0, & n \neq m, \\ 1, & n = m \end{cases}$$

and the associated monic orthogonal polynomials

$$P_n(z) = \kappa_n^{-1} p_n(z).$$

We want to understand how $p_n(z)$ behaves as $n \to \infty$.

# Construction of expansions

Let $\mathcal{M}_{\mathcal{D}}(z,\zeta)$ be some modified Cauchy kernel, meaning that

$$\mathcal{M}_{\mathcal{D}}(z,\zeta) = \frac{\widetilde{\mathcal{M}}_{\mathcal{D}}(z,\zeta)}{\zeta - z}$$

where $\widetilde{\mathcal{M}}_{\mathcal{D}}(z,\zeta)$ is analytic in each variable for $\zeta$ in some annulus $\rho < |\zeta| < 1/\rho$ and $z$ is some open disk $|z| < 1/\rho$, $\rho < 1$.

For instance, we could take $\widetilde{\mathcal{M}}_{\mathcal{D}}(z,\zeta) = A(\zeta)B(z)$ the product of two entire functions.

For every $n \geq 0$, we construct a series of functions as follows:

$$f_{n,0}(z) = 1, \quad z \in \overline{\mathbb{C}}.$$

For every $n \geq 0$, we construct a series of functions as follows:

$$f_{n,0}(z) = 1, \quad z \in \overline{\mathbb{C}}.$$

Assuming the function $f_{n,2k}$ has been defined as an analytic function in $|z| \neq 1$, with analytic boundary values $f_{n,2k}(\zeta)_+$ on the circle unit $\mathbb{T}_1$ from the positive side, we set

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i} \oint_{|\zeta|=1} \mathcal{M}_{\mathcal{D}}(z,\zeta)\, \zeta^n f_{n,2k}(\zeta)_+ \, d\zeta, \quad z \in \mathbb{D}_{1/\rho} \setminus \mathbb{T}_1$$

For every $n \geq 0$, we construct a series of functions as follows:

$$f_{n,0}(z) = 1, \quad z \in \overline{\mathbb{C}}.$$

Assuming the function $f_{n,2k}$ has been defined as an analytic function in $|z| \neq 1$, with analytic boundary values $f_{n,2k}(\zeta)_+$ on the circle unit $\mathbb{T}_1$ from the positive side, we set

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i} \oint_{|\zeta|=1} \mathcal{M}_{\mathcal{D}}(z,\zeta)\, \zeta^n f_{n,2k}(\zeta)_+ \, d\zeta, \quad z \in \mathbb{D}_{1/\rho} \setminus \mathbb{T}_1$$

Assuming the function $f_{n,2k+1}$ has been defined as an analytic function in $\mathbb{D}_{1/\rho} \setminus \mathbb{T}_1$, with analytic boundary values $f_{n,2k+1}(\zeta)_-$ on the circle $\mathbb{T}_1$ from the negative side, we set

$$f_{n,2k+2}(z) := \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\zeta^{-n} f_{n,2k+1}(\zeta)_-}{\zeta - z} d\zeta, \quad |z| \neq 1$$

Then, one can prove that for all $n$ sufficiently large,

$$\mathcal{P}_n(z) := \begin{cases} z^n \displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z), & |z| > 1, \\ -\displaystyle\sum_{k=0}^{\infty} f_{n,2k+1}(z), & |z| < 1, \end{cases}$$

is a monic polynomial of degree $n$, and so is

$$\frac{\mathcal{P}'_{n+1}(z)}{n+1}$$

as well.

Can we choose $\mathcal{M}_{\mathcal{D}}(z, \zeta)$ is such a way that $\mathcal{P}'_{n+1}$ be orthogonal over the CMCD $\mathcal{D}$ with respect to area measure $dA$?

Can we choose $\mathcal{M}_{\mathcal{D}}(z,\zeta)$ is such a way that $\mathcal{P}'_{n+1}$ be orthogonal over the CMCD $\mathcal{D}$ with respect to area measure $dA$?

For that, we need

$$\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) = 0, \quad 0 \leq m \leq n-1,$$

with

$$\mathcal{P}_{n+1}(z) = -\sum_{k=0}^{\infty} f_{n+1,2k+1}(z), \quad |z| < 1$$

Can we choose $\mathcal{M}_{\mathcal{D}}(z,\zeta)$ is such a way that $\mathcal{P}'_{n+1}$ be orthogonal over the CMCD $\mathcal{D}$ with respect to area measure $dA$?

For that, we need

$$\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) = 0, \quad 0 \le m \le n-1,$$

with

$$\mathcal{P}_{n+1}(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_{|\zeta|=1} \mathcal{M}_{\mathcal{D}}(z,\zeta)\,\zeta^{n+1} f_{n+1,2k}(\zeta)_+\,d\zeta, \quad |z|<1$$

Can we choose $\mathcal{M}_{\mathcal{D}}(z,\zeta)$ is such a way that $\mathcal{P}'_{n+1}$ be orthogonal over the CMCD $\mathcal{D}$ with respect to area measure $dA$?

For that, we need

$$\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) = 0, \quad 0 \leq m \leq n-1,$$

with

$$\mathcal{P}_{n+1}(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \mathcal{M}_{\mathcal{D}}(z,\zeta)\, \zeta^{n+1} F_{n+1}(\zeta)_{+}\, d\zeta, \quad |z| < 1$$

with

$$F_{n+1}(z) := \sum_{k=0}^{\infty} f_{n+1,2k}(z)$$

analytic on $|z| \leq 1$.

Can we choose $\mathcal{M}_{\mathcal{D}}(z,\zeta)$ is such a way that $\mathcal{P}'_{n+1}$ be orthogonal over the CMCD $\mathcal{D}$ with respect to area measure $dA$?

For that, we need

$$\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) = 0, \quad 0 \leq m \leq n-1,$$

with

$$\mathcal{P}'_{n+1}(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1}\left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\,\zeta^{n+1}F_{n+1}(\zeta)_{+}\,d\zeta, \quad |z|<1$$

$$
\begin{aligned}
&\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) \\
&= \frac{1}{2\pi i}\int_{\mathcal{D}}\left(\oint_{|\zeta|=1}\left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\zeta^{n+1}F_{n+1}(\zeta)_{+}\ d\zeta\right]\overline{z^m}dA(z) \\
&= \frac{1}{2\pi i}\oint_{|\zeta|=1}\zeta^{n+1}F_{n+1}(\zeta)_{+}\ \left(\int_{\mathcal{D}}\left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\overline{z^m}dA(z)\right)d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z) \\
&= \frac{1}{2\pi i}\int_{\mathcal{D}}\left(\oint_{|\zeta|=1}\left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\zeta^{n+1}F_{n+1}(\zeta)_+\, d\zeta\right]\overline{z^m}dA(z) \\
&= \frac{1}{2\pi i}\oint_{|\zeta|=1}\zeta^{n+1}F_{n+1}(\zeta)_+\,\left(\int_{\mathcal{D}}\left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\overline{z^m}dA(z)\right)d\zeta
\end{aligned}
$$

If we now choose

$$
\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta) = \frac{\mathcal{K}_{\mathcal{D}}(z,1/\overline{\zeta})}{\zeta^2}
$$

where

$$
\mathcal{K}_{\mathcal{D}}(z,\zeta) = \sum_{k=0}^{\infty} p_n(z)\overline{p_n(\zeta)}
$$

$$
\begin{aligned}
&\int_{\mathcal{D}} \mathcal{P}'_{n+1}(z)\overline{z^m}dA(z)\\
&= \frac{1}{2\pi i}\oint_{|\zeta|=1} \zeta^{n+1}F_{n+1}(\zeta)_+ \left(\int_{\mathcal{D}} \left[\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta)\right]\overline{z^m}dA(z)\right)d\zeta\\
&= \frac{1}{2\pi i}\oint_{|\zeta|=1} \zeta^{n-1}F_{n+1}(\zeta)_+ \underbrace{\left(\int_{\mathcal{D}} \mathcal{K}_{\mathcal{D}}(z,1/\overline{\zeta})\overline{z^m}dA(z)\right)}_{\zeta^{-m}}d\zeta
\end{aligned}
$$

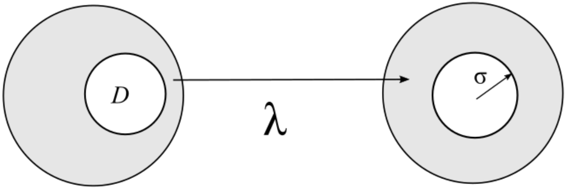
$$
= -\frac{1}{2\pi i}\oint_{|\zeta|=1} \zeta^{n-m-1}F_{n+1}(\zeta)_+ d\zeta = \begin{cases} 0, & 0 \le m \le n-1 \\ \pi F_{n+1}(0), & m = n. \end{cases}
$$

Then the choice

$$\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta) = \frac{\mathcal{K}_{\mathcal{D}}(z, 1/\overline{\zeta})}{\zeta^2}$$

will do. Now our task is to find

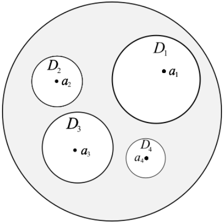
$$\mathcal{K}_{\mathcal{D}}(z,\zeta) = \sum_{k=0}^{\infty} p_n(z)\overline{p_n(\zeta)}$$




There is a unique automorphism $\lambda$ of the unit disk of the form

$$\lambda(z) = \frac{z - a}{1 - z\bar{a}}$$

such that $\lambda(D)$ is a disk centered at the origin.




For each $k \in \{1, 2, \ldots, N\}$, let

- $$\lambda_k(z) = \frac{z - a_k}{1 - \overline{a_k} z}$$

  be such that $\lambda_k(D_k)$ is a disk centered at the origin.
- $\sigma_k$ denotes the radius of $\lambda_k(D_k)$, and
- $\lambda_k^{-1}$ denotes the inverse of $\lambda_k$.

For each $k \in \{1, 2, \ldots, N\}$, define the transformation

$$T_k(z) := \lambda_k^{-1}[\sigma_k^2 \lambda_k(z)], \quad z \in \overline{\mathbb{C}}.$$

For each $k \in \{1, 2, \dots, N\}$, define the transformation

$$T_k(z) := \lambda_k^{-1}[\sigma_k^2 \lambda_k(z)], \quad z \in \overline{\mathbb{C}}.$$

Let $\mathcal{T}$ denote the family of functions $\gamma$ of the form

$$\tau = T_{k_n} T_{k_{n-1}} \cdots T_{k_2} T_{k_1}, \quad 1 \le k_i \le N, \ n \in \mathbb{N},$$

together with $\tau_0(z) = z$.

Here, each function $\tau$ as above is a Möbius transformation that takes $\mathbb{D}$ into $D_{k_n}$.
Let

$$\mathcal{T}_k = \{\tau : \tau = T_k \cdots\}, \qquad k = 1, \dots, N.$$

Let

$$\rho := \max\{|a_k| : 1 \leq k \leq N\}$$

## Theorem

*The series* $\sum_{\tau \in \mathcal{T}} |\tau'(z)|$ *converges for all* $|z| < 1/\rho$ *and*

$$\mathcal{K}_{\mathcal{D}}(z,\zeta) = \sum_{\tau \in \mathcal{T}} \frac{\tau'(z)}{[1 - \tau(z)\overline{\zeta}]^2}, \quad z,\zeta \in \mathbb{D}_1.$$

Then

$$\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta) = \frac{\mathcal{K}_{\mathcal{D}}(z, 1/\overline{\zeta})}{\zeta^2}$$

can be accomplished by choosing, for instance,

$$\mathcal{M}_{\mathcal{D}}(z,\zeta) := \frac{1}{\zeta}\cdot\frac{z}{\zeta - z} + \sum_{\tau\in\mathcal{T}^*}\left[\frac{1}{\zeta-\tau(0)}\cdot\frac{\tau(z)-\tau(0)}{\zeta-\tau(z)}\right].$$

where

$$\mathcal{T}^* = \mathcal{T}\setminus\{\tau_0\}$$

Then, after some work using the series expansions, we can get the following results.

Let

$$\rho := \max\{|a_k| : 1 \leq k \leq N\}$$

## Theorem

*For n sufficiently large, we have*

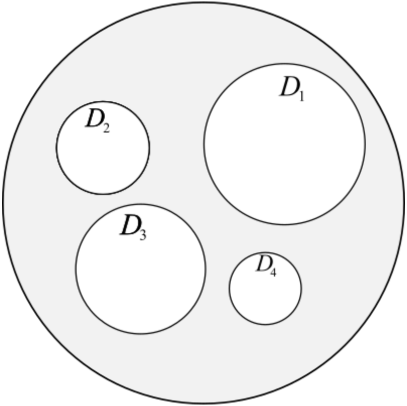
$$P_n(z) = \sum_{\tau \in \mathfrak{T}} [\tau(z)]^n \tau'(z)[1 + o(1)], \quad z \in \mathbb{D}_{1/\rho},$$

*locally uniformly as $n \to \infty$ in $\mathbb{D}_{1/\rho}$. Equivalently,*

$$P_n(z) = z^n \cdot [1 + o(1)] + \sum_{j=1}^{N} P_n(T_j(z)) T_j'(z), \quad z \in \mathbb{D}_{1/\rho}.$$

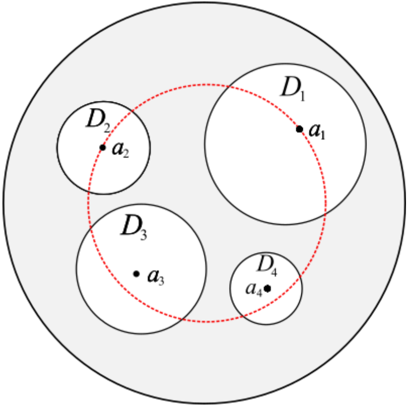
The behavior of $P_n$ is mostly determined by those $a_k$ with largest modulus. Suppose

$$\rho = |a_1| = |a_2| = \cdots = |a_s| > |a_{s+1}| \geq \cdots \geq |a_N|$$



The behavior of $P_n$ is mostly determined by those $a_k$ with largest modulus. Suppose

$$\rho = |a_1| = |a_2| = \cdots = |a_s| > |a_{s+1}| \geq \cdots \geq |a_N|$$



## Theorem

(i) *For all* $r \in (\rho, \rho^{-1})$,

$$\frac{P_n(z)}{z^n} = 1 + O(\eta_r^n), \quad |z| \geq r,$$

*where*

$$\eta_r := r^{-1} \max_{1 \leq j \leq s, |z|=r} |T_j(z)| < 1.$$

(ii) *When* $|z| = \rho$ *but* $z \notin \{a_1, \ldots, a_s\}$,

$$\frac{P_n(z)}{z^n} = 1 + O\left(\frac{1}{n}\right),$$

*while for* $z = a_j$, $1 \leq j \leq s$,

$$\frac{P_n(a_j)}{a_j^n} = \frac{1}{1 - \sigma_j^2} + O\left(\frac{1}{n}\right)$$

For the asymptotics on $|z| < \rho$, we need a few items:

$$\theta_j := \operatorname{Arg} a_j, \quad \beta_j := \frac{1}{\overline{a_j}} - a_j, \quad \mathcal{T}_j := \{T_j\tau : \tau \in \mathcal{T}\}$$

$$\mathcal{H} := \{z : \operatorname{Re}(z) < 0\}$$

For each $1 \leq j \leq s$, we define

$$\Theta_j(t) := t \sum_{v \in \mathbb{Z}} \sigma_j^{2v} \exp(\overline{\beta_j} \sigma_j^{2v} t), \quad t \in e^{i\theta_j}\mathcal{H}$$

and

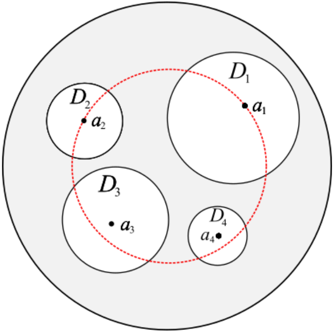
$$\mathcal{J}_{j,n}(z) := -\sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \frac{\lambda_j'(\tau(z))}{\lambda_j(\tau(z))} \Theta_j(n\lambda_j(\tau(z)))\tau'(z), \quad z \in \mathbb{D}_\rho.$$

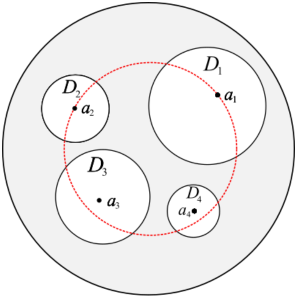
The functions $\mathcal{J}_{j,n}$ are bounded on compacts of $\mathbb{D}_\rho$.

## Theorem

*Uniformly as $n \to \infty$ on compacts of $|z| < \rho$,*

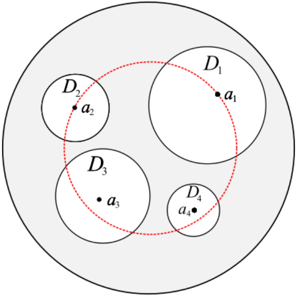
$$\frac{nP_n(z)}{\rho^n} = (1-\rho^2)\sum_{j=1}^{s} e^{in\theta_j}\mathcal{J}_{j,n}(z) + O\left(\frac{1}{n}\right)$$






In the case $s = 1$ of just one disk removed, say with $a_1 > 0$, the formula takes the shape

$$\frac{nP_n(z)}{a_1^n} = -(1 - a_1^2)\frac{\lambda_1'(z)}{\lambda_1(z)}\Theta_1(n\lambda_1(z)) + O\left(\frac{1}{n}\right),$$




In the case $s = 1$ of just one disk removed, say with $a_1 > 0$, the formula takes the shape

$$\frac{nP_n(z)}{a_1^n} = -(1 - a_1^2)\frac{\lambda_1'(z)}{\lambda_1(z)}\Theta_1(n\lambda_1(z)) + O\left(\frac{1}{n}\right),$$

$$\Theta_1(t) = t\sum_{v\in\mathbb{Z}} \sigma_1^{2v} \exp(\overline{\beta_1}\sigma_1^{2v} t), \qquad \lambda_1(z) = \frac{z - a_1}{1 - \overline{a_1}z}$$

Given a subsequence $\{n_k\} \subset \mathbb{N}$, the sequence $\{\Theta_1(n_k t)\}$ converges normally on $\mathrm{Re}(t) < 0$ if and only if

$$\lim_{k\to\infty} \langle \log_{\sigma_1^2} n_k \rangle = q \in [0,1),$$

in which case

$$\lim_{k\to\infty} \Theta_1(n_k t) = \Theta_1(\sigma_1^{2q} t)$$

and thus by the asymptotic representation,

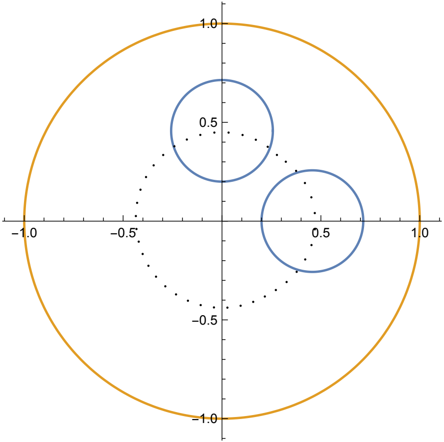
$$\lim_{k\to\infty} \frac{n_k P_{n_k}(z)}{a_1^{n_k}} = -\frac{\lambda_1'(z)(1-a_1^2)}{\lambda_1(z)} \Theta_1(\sigma_1^{2q} \lambda_1(z)).$$

Thus, the sequence $\{na_1^{-n}P_n(z)\}$ has for normal limit points on $|z| < a_1$ the continuous one-parameter family of functions

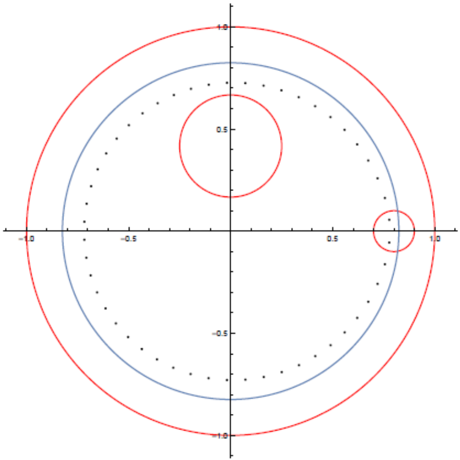
$$\left\{-\frac{\lambda'(z)(1-a_1^2)}{\lambda_1(z)}\Theta_1(\sigma_1^{2q}\lambda_1(z)) : \ q \in [0,1)\right\}.$$

How about the zeros of $P_n$?

Most of them will accumulate toward the circle $|z| = \rho$.




Zeros of $P_n$ for $n = 40$




Zeros of $P_n$ for $n = 50$

Thank you!