Behaviour of multipoled pluricomplex Green's fts in connection with algebraic geometry pbs

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Nagata Conjecture in $\mathbb{P}^2$

The Nagata Conjecture is one of the most intriguing open problems in the area of curves in the projective plane. It governs the minimal degree required for a plane algebraic curve to pass through a collection of r general points in $\mathbb{P}^2$ with prescribed multiplicities.

Suppose $p_1, \ldots, p_r$ are r general points in $\mathbb{P}^2$ and that $m_1, \ldots, m_r$ are given positive integers. Then for $r > 9$, any curve C in $\mathbb{P}^2$ that passes through each of the points p_i with multiplicity m_i must satisfy deg $C > \frac{1}{\sqrt{r}} . \sum_{i=1}^{r} m_i$.

One says that a property $\mathcal{P}$ holds for r general points in $\mathbb{P}^2$ if there is a non empty Zariski-open subset W of $(\mathbb{P}^2)^r$ such that $\mathcal{P}$ holds for every "set" $\{p_1, \ldots, p_r\}$ of r points in W.

Masayoshi Nagata (1959) pointed out that it is enough to consider the uniform case. Thus this conjecture is equivalent to the following one, which is usually called:

Nagata Conjecture.

Suppose $p_1, \ldots, p_r$ are r general points in $\mathbb{P}^2$ and that m is a given positive integer. Then for $r > 9$, any curve C in $\mathbb{P}^2$ that passes through each of the points p_i with multiplicity at least m must satisfy $\deg C > \sqrt{r}.m$.

The only case when this is known to hold true is when $r > 9$ is a perfect square (proved by M. Nagata 59).

He has also remarked that the condition $r > 9$ is necessary.

Fix r general points $S = \{p_1, \ldots, p_r\}$ in $\mathbb{P}^2$ and a nonnegative integer m. Define $\Omega(S, m)$ to be the least integer d such that there is a curve of degree d vanishing at each point p_i with multiplicity at least m. For $r \leq 9$, applying M. Nagata's methods, B. Harbourne (01) shows that $\Omega(S, m) = \lceil c_r m \rceil$, where

r	1	2	3	4	5	6	7	8	9
$c_r = \inf\limits_{m \geq 1} \dfrac{\Omega(S,m)}{m}$	1	1	$\frac{3}{2}$	2	2	$\frac{12}{5}$	$\frac{21}{8}$	$\frac{48}{17}$	3

More recently, without any extra condition connecting the degree, the multiplicity m and the cardinality r,
G. Xu (94) proved that $\deg C \geq \frac{\sqrt{r-1}}{r} \cdot \sum_{i=1}^{r} m_i$
and H. Tutaj-Gasinska (03) proved that $\deg C \geq \frac{1}{\sqrt{r+(1/12)}} \cdot \sum_{i=1}^{r} m_i$
(see also B. Harbourne (01) and B. Harbourne and J. Roé (08)).

An affirmative answer to Nagata conjecture is very important because it has a lot of applications in particular in number theory, in symplectic geometry (symplectic packings in the unit ball) and in algebraic geometry.
A more modern formulation of this conjecture is often given in terms of multiple-point Seshadri constants, introduced by J.-P. Demailly (92-01) in the course of his work on Fujita's conjecture.

Generalization of Nagata Conj. in any dimension $n \geq 2$

Based on a conjecture of R. Fröberg, A. Iarrobino (97) predicted that

An hypersurface in $\mathbb{P}^n$ passing through r generic points with multiplicity m has a degree $> r^{1/n} \cdot m$, except for an explicit finite list of (r, n).

L. Evain (05) proved this conjecture when the number of points in $\mathbb{P}^n$ is of the form s^n (in this case the list is $(4, 2)$, $(9, 2)$ and $(8, 3)$).

Up to now, Nagata Conjecture remains open for every non-square $r \geq 10$, after more than sixty years of attention by many researchers.
There are over 200 references in Mathscinet, directly on this subject.

SHGH Conjecture in $\mathbb{C}^2$

Fix r general points $S = \{p_1, \ldots, p_r\}$ in $\mathbb{C}^2$ and nonnegative integers d and m. We denote by

$\mathcal{P}_S(d;m)$: the linear subspace of polynomials of degree $\leq d$ in $\mathbb{C}^2$, with vanishing order at least m at any points p_i, for $1 \leq i \leq r$.

The expected dimension of $\mathcal{P}_S(d;m)$ is $e_S(d;m) = \max\{0, v_S(d;m)\}$

where $v_S(d;m) = \frac{(d+1)(d+2)}{2} - r\frac{m(m+1)}{2}$ is its virtual dimension.

The true dimension of the space $\mathcal{P}_S(d;m)$ is denoted by $\dim_S(d;m)$ and

$$\dim_S(d;m) \geq e_S(d;m).$$

B. Harbourne (86), A. Gimigliano (87) and A. Hirschowitz (89), independently have given conjectures for explicitly computing $\dim_S(d;m)$ for any r. Those conjectures are in connection with an earlier conjecture posed by B. Segre (62), and they are all equivalent to the following one, which states

SHGH Conjecture.

Suppose that $S = \{p_1, \ldots, p_r\}$ are r general points in $\mathbb{C}^2$ and that m is a given positive integer. Then $\dim_S(d;m) = e_S(d;m)$

This last conjecture is an interpolation problem. It is obvious in $\mathbb{C}$, but difficult if the number of variables is greater than or equal to 2.

SHGH Conjecture has been proved independently by C. Ciliberto and R. Miranda (06) and L. Evain (07), when r is a perfect square.

Connection between these two Conjectures

SHGH conjecture $\Longrightarrow$ Nagata conjecture.

If the SHGH Conjecture is satisfied, the minimal degree of a curve through $r \geq 10$ points with vanishing order $\geq m$, is always $> \sqrt{r}.m$ (because $\frac{(d+1)(d+2)}{2} \geq r\frac{m(m+1)}{2} + 1$),
and when m goes to ∞ it is asymptotically equal to $\sqrt{r}.m + \lfloor \frac{\sqrt{r}-3}{2} \rfloor$.

Nagata and SHGH Conjectures in connection with interpolation problem

Fix $S = \{p_1, \ldots, p_r\}$, $d \geq 1$ and $m \geq 1$. Let denote by
* $\mathbb{C}^{2r}_* := \{z = (z_1, \ldots, z_r) \in \mathbb{C}^{2r} : z_j \neq z_k, \forall j \neq k\}$,
* $\delta_1(r,m) := r\frac{m(m+1)}{2}$, $d_1(r,m) = \lfloor \sqrt{r}.m \rfloor$,
* $\varphi_{S,d,m}$ the evaluation map from $\mathbb{C}_d[z]$ to $\mathbb{C}^{\delta_1(r,m)}$

$$P \mapsto (P^{(\gamma)}(s), \text{ for all } |\gamma| \leq m-1 \text{ and for all } p_j \in S).$$

S is a "Nagata point in $\mathbb{C}^{2r}_*$" iff for any $m \geq 1$, the map $\varphi_{S,d_1(r,l),m}$ is injective (or equivalently, has maximal rank).

When r is a square ≥ 10, there exists "Nagata points".
More precisely there exists a psh function w_r in the Lelong class in $\mathbb{C}^{2r}$ (not $\equiv -\infty$) such that

$$\mathbb{C}^{2r}_* \setminus w_r^{-1}(-\infty) \subset \mathrm{N}ag_r.$$

S is a "SHGH point in $\mathbb{C}_*^{2r}$" iff for any $m \geq 1$ and any $d \geq 1$, $\varphi_{S,d,m}$ **has maximal rank**:

$*$ when $\frac{(d+1)(d+2)}{2} \leq r\frac{m(m+1)}{2}$, $\varphi_{S,d,m}$ is injective,
$*$ when $\frac{(d+1)(d+2)}{2} \geq r\frac{m(m+1)}{2}$, $\varphi_{S,d,m}$ is surjective.

When r is a square ≥ 10, there exists SHGH points.
More precisely there exists a psh function $\tilde{w}_r$ in the Lelong class in $\mathbb{C}^{2r}$ (not $\equiv -\infty$) such that

$$\mathbb{C}_*^{2r} \setminus \tilde{w}_r^{-1}(-\infty) \subset \mathrm{S}HGH_r.$$

Transcendental versions of these conjectures in term of pluripotential theory

Now we would like to develop transcendental techniques, to overcome the intrinsic rigidity of polynomials and to obtain a new approach to this problem of algebraic geometry. Instead of considering complex polynomials, we work with plurisubharmonic functions, having logarithmic poles at prescribed points. These last functions are much more flexible than the first ones.

Thus two points of view are possible. A local one and a global one.

- Pluricomplex Green functions in a bounded domain in $\mathbb{C}^n$ with logarithmic poles of weight 1 at prescribed points with zero value at the boundary.

  Q1. What happens when the poles collide to a single point in the domain ? The nature of the logarithmic singularity of the limit function is in connection with the algebraic properties of the set of fixed points and its singular degree of M. Waldschmidt (77-87).

- Entire psh functions in all $\mathbb{C}^n$, with a logarithmic behavior at infinity. In particular the subclass of psh functions which are maximal (for the complex Monge-Ampère operator) outside prescribed points. They inevitably have to satisfy certain conditions of growth at infinity.

  Q2. Which minimal growth they can have at infinity ?
  It will have also a link with the singular degree of M.Waldschmidt.

Recalls "Pluricomplex Green function in bounded domain in $\mathbb{C}^n$"

For a bounded domain $D \subset \mathbb{C}^n$, the *pluricomplex Green function in D with logarithmic poles in a finite subset S of D*

$$g_D(S, z) = \sup\{u(z) : u \text{ psh on } D,\ u \leq 0,$$

$$u(z) \leq \ln ||z - p|| + O(1) \text{ for any point } p \text{ in } S\}.$$

(J.-P. Demailly (85-87), M. Klimek (87), P. Lelong (87-89)).
If D is hyperconvex then this ft has an alternative description in terms of the complex Monge-Ampère operator (E. Bedford- B.-A. Taylor (76-82), J.-P. Demailly(87)). $g_D(S, .)$ is the unique solution to the following Dirichlet pb:

$$\begin{cases} u \text{ is plurisubharmonic and negative on } D, \text{ continuous on } \bar{D} \setminus S, \\ (dd^c u)^n = 0 \text{ on } D \setminus S, \\ u(z) = \ln ||z - p|| + O(1) \text{ as } z \to p, \forall p \in S, \\ u(z) \to 0 \text{ as } z \to \partial D. \end{cases}$$

In this case, $(dd^c u)^n = \sum_{p \in S} \delta_p$.

Recalls "Singular degree of a finite set of points"

It's an affine invariant, introduced by M. Waldschmidt (87) (G.V. Chudnovsky (81)), in connection with the Nagata Conjecture.

For any polynomial $P \in \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$,
$\deg P$ its degree and $\mathrm{ord}(P, p)$ its vanishing order at any point p.
If m is a positive integer

$$\Omega(S, m) = \min\{\text{deg P} : P \in \mathbb{C}[z], \mathrm{ord}(P, p) \geq m, \forall\, p \in S\}.$$

The limit exists and it is called the singular degree of S

$$\Omega(S) := \lim_{m \to +\infty} \Omega(S, m)/m = \inf_{m \geq 1} \Omega(S, m)/m$$

$(\Omega(S, m_1 + m_2) \leq \Omega(S, m_1) + \Omega(S, m_2)$, and in particular $\Omega(S, m) \leq \Omega(S, 1)m)$.

For any $m \geq 1$

$$\frac{\Omega(S,1)}{n} \leq \Omega(S) = \inf_m \frac{\Omega(S,m)}{m} \leq \frac{\Omega(S,m)}{m} \leq \Omega(S,1).$$

By using Hörmander-Bombieri-Skoda theorem, M. Waldschmidt (87) proved

$*$ for any m_1 and $m_2 \geq 1$: $\dfrac{\Omega(S,m_1)}{m_1+n-1} \leq \Omega(S) \leq \dfrac{\Omega(S,m_2)}{m_2}$,

$*$ an upper bound for $\Omega(S,m)$:

$$\Omega(S,m) \leq (m+n-1)|S|^{1/n} - (n-1).$$

$\Longrightarrow \Omega(S) \leq |S|^{1/n}$.

The problem is to find a lower bound for $\Omega(S)$. Nagata conjecture can be stated in term of the invariants $\Omega(S,m)$:

In $\mathbb{C}^2$, if $r > 9$, then $[\Omega(S,m) > m\sqrt{r}, \forall\, m \geq 1]$ holds for a set S of r points in general position.

One application: a Schwarz' Lemma (Moreau (80), Waldschmidt (87))

For any $\epsilon > 0$, there exists a real number $r(S, \epsilon)$ s.t.
for any $r < \varrho < R/(2e^n)$ and
for any entire function f s.t. $\mathrm{ord}(f, p_j) \leq m$ for all $p_j \in S$, we have

$$\begin{aligned} \ln ||f||_\varrho - \ln ||f||_R \ &\leq \ (\Omega(S, l) - l\epsilon). \ln\left(\frac{2e^n\varrho}{R}\right) \\ &\leq l(\Omega(S) - \epsilon) \ln\left(\frac{2e^n\varrho}{R}\right). \end{aligned}$$

Local study

Goal. Study the convergence of multipole pluricomplex Green functions in a bounded hyperconvex domain in $\mathbb{C}^n$, in the case where poles contract to one single point.

Reduce this pb to the case where the domain $= B(O,1)$ and the point where the poles contract is the origin.
Let $S \subset B(O,1)$. $|S|$ its cardinality.
$g_R(S,.)$ the pluricomplex Green function in ball $B(O,R)$ with logarithmic poles in S, of weight one.
$g_R(S,z) = g_1(S/R, z/R)$ for any $z \in B(O,R)$.
Thus its is natural to study g_∞ a negative psh function defined in $B(O,1)$ by

$$g_\infty(z) = (\limsup_{t \in \mathbb{C}^* \to 0} g_1(tS,z))^*.$$

$$|S| \cdot \ln ||z|| \leq g_\infty(z) \leq g_{B(O,1)}(O,z) = \ln ||z||.$$

g_∞ tends to 0 on the boundary of $B(O,1)$ and it has an unique logarithmic singularity at the origin.

Q. What is the nature of the logarithmic singularity of this function g_∞ at O ? What is its Lelong number at O ?

If u is a psh function, then the *classical Lelong number* $\nu(u,z)$ of u at a point z is (P. Lelong-69) the $(2n-2)$-dimensional density of the measure $dd^c u$ at z :

$$\nu(u,z) := \lim_{r\to 0} \frac{1}{(\pi r^2)^{n-1}} \int_{|w-z|<r} dd^c u \wedge (dd^c|w-z|^2)^{n-1}.$$

We can also compute this number as follows (Avanissian, K. Kiselman):

$$\nu(u,z) = \lim_{y\to-\infty} \frac{\sup_{|w|=1} u(z+we^y)}{y} = \lim_{y\to-\infty} \frac{1}{y} \int_{|w|=1} u(z+we^y)\, d\tilde{\lambda}(w),$$

where $d\tilde{\lambda}$ is the normalized surface measure on the unit sphere.

Theorem (N-AIF 21)

Let S be a finite set of points in $\mathbb{C}^n$. The psh function g_∞ satisfy several properties :

1. $\nu(g_\infty, O) = \Omega(S)$ and $(dd^c g_\infty)^n = 0$ in $B(O,1) \setminus \{O\}$.
   $\Omega(S)^n \leq (dd^c g_\infty)^n(\{O\}) = \int_{B(O,1)} (dd^c g_\infty)^n \leq |S|$ and we have
   $$g_\infty(z) \leq \Omega(S) \ln ||z||, \text{ in } B(O,1).$$
2. If $\Omega(S) = |S|^{1/n}$ then $(dd^c g_\infty)^n(\{O\}) = \Omega(S)^n = |S|$ and
   $$g_\infty(z) = |S|^{1/n} \ln ||z||, \quad \text{in } B(O,1).$$
3. Conversely if g_∞ is equal to $\Omega(S) \ln ||.||$ in $\bar{B}(O,1)$, then $\Omega(S) = |S|^{1/n}$.

In the previous construction of g_∞, we can replace the unit ball $B(O,1)$ by any bounded hyperconvex domain D in $\mathbb{C}^n$ and the origin by any point z_o in D.

2 examples with two or three poles

1. Let $S = \{(1/2,0), (-1/2,0)\}$ in $\mathbb{C}^2$. $\Omega(S) = 1$. For any $t \in \mathbb{C}^*$ suff. small,

$$g_1(tS,z) = \max\{\ln\left|\frac{(z_1 - t/2)(z_1 + t/2)}{(1-\bar{t}z_1/2)(1+\bar{t}z_1/2)}\right|, \ln|z_2|\}, \text{ in } P(O,1)$$

$(g_1(tS,.))_{t\in\mathbb{C}^*}$ cv. loc. unif. outside O to $g_\infty(z) = \max\{2\ln|z_1|, \ln|z_2|\}$.
$(dd^c g_\infty)^2(\{O\}) = 2 = |S|$ and $\nu(g_\infty, O) = 1 < \sqrt{2}$.

2. Let $S = \{(0,0), (1,0), (0,1)\}$ in $\mathbb{C}^2$. $\Omega(S) = 3/2$. Around the origin

$$g_1(tS,z) = \max\{\ln|z_1z_2|, \ln|z_1(z_1+z_2-t)|, \ln|z_2(z_1+z_2-t)|, \\ \tfrac{1}{2}\ln|z_1z_2(z_1+z_2-t)|\} + O(1).$$

$(g_1(tS,.))_{t\in\mathbb{C}^*}$ cv. loc. unif. outside O to $g_\infty(z) =$
$\max\{\ln|z_1z_2|, \ln|z_1(z_1+z_2)|, \ln|z_2(z_1+z_2)|, \tfrac{1}{2}\ln|z_1z_2(z_1+z_2)|\} + O(1)$.

$$(dd^c g_\infty)^2(\{O\}) = 2.\frac{3}{2} = 3 = |S| \text{ and } \nu(g_\infty, O) = \frac{3}{2} < \sqrt{3}.$$

Conjecture ($\mathcal{P}_1$)

In $\mathbb{C}^n$, except for a finite number of integer values of r, for any general set $S = \{p_1, \ldots, p_r\}$ of r points, the family of pluricomplex Green functions $(g_{B(O,1)}(tS, .))_{t \in \mathbb{C}^*}$ converges locally uniformly outside the origin of $B(O, 1)$ to $r^{1/n} g_{B(O,1)}(O, .)$, when t tends to 0.

$$\Updownarrow$$

Conjecture ($\mathcal{A}_1$). Weak Version of Nagata Conjecture in $\mathbb{P}^n$

In $\mathbb{P}^n$, except for a finite number of integer values of r, for any general set $S = \{p_1, \ldots, p_r\}$ of r points, $\Omega(S) = r^{1/n}$.

Global study

Now we consider a class of entire psh functions in $\mathbb{C}^n$, with logarithmic poles in a finite set of points S and with a logarithmic growth at infinity; and in particular, the subclass of such functions which are also locally bounded outside of S.

If u is a psh function in $\mathbb{C}^n$

$$\gamma_u := \limsup_{|z|\to\infty} \frac{u(z)}{\log ||z||} \in [0, +\infty].$$

Two affine invariants

$$\Omega_{psh}(S) := \inf\{\gamma_u : u \in PSH(\mathbb{C}^n),\ \nu(u,p) \geq 1 \text{ for any } p \in S\}.$$

$$\Omega^+_{psh}(S) := \inf\{\gamma_u : u \in PSH(\mathbb{C}^n) \cap L^\infty_{loc}(\mathbb{C}^n \setminus S),\ \nu(u,p) \geq 1 \text{ for any } p \in S\}.$$

(this last one has been introduced by Coman-Nivoche (02)).

We can state two more conjectures $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$.
$(\mathcal{P}_2)$ can be seen as the dual version of the first one $(\mathcal{P}_1)$.

Conjecture $(\mathcal{P}_2)$.

In $\mathbb{C}^n$, except for a finite number of integer values of r, for any general set S of r points, we have : for any $\epsilon > 0$, there exists an entire continuous psh function v in $L^\infty_{loc}(\mathbb{C}^n \setminus S)$, such that $\nu(v,p) \geq 1$ for any $p \in S$ any $\gamma_v \leq (1+\epsilon)|S|^{1/n}$.

Conjecture $(\mathcal{P}_3)$.

In $\mathbb{C}^n$, except for a finite number of integer values of r, for any general set S of r points, we have : $\Omega_{psh}(S) = \Omega^+_{psh}(S)$.

Theorem (N-AIF 21)

$$(\mathcal{P}_1) \Leftrightarrow (\mathcal{P}_2) \Leftrightarrow (\mathcal{P}_3)$$

To prove this, we use a **comparison principle** which relates Lelong numbers at the points of S of two psh functions with their logarithmic growth at infinity (the proof of this result is similar to a result in D. Coman-S. Nivoche-02, D. Coman-06).

Proposition (N-AIF 21)

Let $S \subset \mathbb{C}^n$ be a finite set. Let u and v be two psh functions in $\mathbb{C}^n$ s.t. $u \in L^\infty_{loc}(\mathbb{C}^n \setminus S)$. Then

$$\sum_{p \in S} \nu(u,p)^{n-1} \nu(v,p) \leq \gamma_u^{n-1} \gamma_v.$$

Nagata Conj. is "almost" equivalent to SHGH Conj.

Fix S in $\mathbb{C}_*^{2r} := \{z = (z_1, \ldots, z_r) \in \mathbb{C}^{2r} : z_j \neq z_k, \forall j \neq k\}$.
If S is a "Nagata point in $\mathbb{C}_*^{2r}$", then for any $l \geq 1$ and $d \geq 1$ such that $\mathrm{Ker}(\varphi_{S,d,l})$ isn't reduced to $\{0\}$, then its dimension satisfies

$$\max\{1, \frac{(d+1)(d+2)}{2} - r\frac{l(l+1)}{2}\} \leq \dim \mathrm{Ker}(\varphi_{S,d,l})$$

$$\leq \frac{(d+1)(d+2)}{2} - \frac{(\lfloor\sqrt{r}.l\rfloor + 1)(\lfloor\sqrt{r}.l\rfloor + 2)}{2}.$$

Northern Cardinal

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Thank you for your attention !