Electrostatic models for orthogonal and multiple orthogonal polynomials.

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Joint work with **A. Martínez–Finkelshtein** (Baylor Univ., TX, USA, and Univ. Almería, Spain) and **Joaquín Sánchez Lara** (Univ. Granada, Spain).

Zeros of Jacobi polynomials: Electrostatic approach

Thomas Jan Stieltjes (1856-1894)

Electrostatic interpretation of zeros of Jacobi Polynomials

Carl Gustav Jacobi (1804-1851)

Zeros of Jacobi polynomials: Electrostatic approach

Jacobi polynomials

$$\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}\,(1+x)^{\beta}\,,\ \alpha,\beta > -1$$

$$\int_{-1}^{1} x^k\, P_n^{(\alpha,\beta)}(x)\,\omega^{(\alpha,\beta)}(x)\,dx \;=\; 0\,,\ k = 0,\ldots,n-1$$

$$P_n^{(\alpha,\beta)}(z) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(z-1)^k(z+1)^{n-k}\,,$$

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!}(z-1)^{-\alpha}(z+1)^{-\beta}\left(\frac{d}{dz}\right)^n\left[(z-1)^{n+\alpha}(z+1)^{n+\beta}\right]\,,$$

for $(\alpha,\beta)\in\mathbb{C}$.

Zeros of Jacobi polynomials: Electrostatic approach

n free unit charges in $(-1,+1)$

Two positive charges at the endpoints: a in $+1$ and b in -1

Charges interact according to the logarithmic potential

Equilibrium problem: To find the positions $x_1, \ldots, x_n$ for the free charges in order to minimize the (logarithmic) energy of the system

$$E(x_1, \ldots, x_n) = -\sum_{i<j} \ln|x_i - x_j| - a\sum_{i=1}^{n} \ln|1 - x_i| - b\sum_{i=1}^{n} \ln|1 + x_i|$$

Zeros of Jacobi polynomials: Electrostatic approach

$E(x_1, \ldots, x_n)$ attains a global minimum on the simplex $-1 \le x_1 \le \ldots \le x_n \le 1$. This minimum is attained in an "inner" point: $-1 < x_1^* < x_2^* < \ldots < x_n^* < +1$

$$\frac{\partial E}{\partial x_k}(x_1^*, \ldots, x_n^*) = 0, \; k = 1, \ldots, n$$

$$\sum_{j \neq k} \frac{1}{x_k^* - x_j^*} + \frac{a}{x_k^* - 1} + \frac{b}{x_k^* + 1} = 0 \,, \; k = 1, \ldots, n \,,$$

$$Q_n(x) = \prod_{k=1}^{n} (x - x_k^*) \,,$$

$$\frac{1}{2} \frac{Q_n''(x_k^*)}{Q_n'(x_k^*)} + \frac{a}{x_k^* - 1} + \frac{b}{x_k^* + 1} = 0 \,, \; k = 1, \ldots, n \,,$$

Zeros of Jacobi polynomials: Electrostatic approach

Polynomial $(x^2-1)Q_n''(x)+2\left(a(x+1)+b(x-1)\right)Q_n'(x)$, of degree $\leq n$, has the same zeros as Q_n, that is:

$$(x^2-1)Q_n''(x)+2\left(a(x+1)+b(x-1)\right)Q_n'(x)=\lambda_n Q_n(x)$$

Jacobi differential equation:

$$(x^2-1)y''(x)+[-\beta+\alpha+(\alpha+\beta+2)x]y'(x)=n(n+\alpha+\beta+1)y(x)$$

$\Rightarrow$ The minimum energy takes place when free charges are located on zeros of the Jacobi polynomial $Q_n = P_n^{(\alpha,\beta)}$,
with $\alpha = 2a-1$ and $\beta = 2b-1 \implies$

$$a=\frac{\alpha+1}{2}\,,\; b=\frac{\beta+1}{2}$$

Extension: Unbounded intervals

Laguerre polynomials: $L_n^{(\alpha)}(x)$

$$\omega^{(\alpha)}(x) = x^{\alpha}\, e^{-x}\,,\; x \in (0,\infty)\,,\; \alpha > -1$$

n positive unit charges in $[0, +\infty)$,
a positive charge p at the origin.

Additional restriction:

$$\sum_{k=1}^{n} x_k \leq Kn \;\rightarrow\; \sum_{k=1}^{n} x_k = Kn \;\rightarrow\; \text{Lagrange multipliers...}$$

Modern viewpoint (Ismail, 2000 ...): Action of the external field

$$\varphi(x) = x, x \in [0, \infty)$$

Extension: Unbounded intervals

Hermite polynomials: $H_n(x)$

$$\omega(x) = e^{-x^2}, x \in (-\infty, \infty)$$

n positive unit charges in $\mathbb{R}$.

Additional restriction:

$$\sum_{k=1}^{n} x_k^2 \leq Kn \rightarrow \sum_{k=1}^{n} x_k^2 = Kn \rightarrow \text{ Lagrange multipliers...}$$

Modern viewpoint (Ismail, 2000 ...): Action of the external field

$$\varphi(x) = \frac{x^2}{2}, x \in \mathbb{R}$$

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

Simple example: Jacobi-Angelesco Polynomials.
V. Kaliaguine (1979); V. Kaliaguine, A. Ronveaux (1996).
Polynomials $P_{n,n} \in \mathbb{P}_{2n}$ satisfying the following system of orthogonality conditions:

$$\int_0^1 x^k P_{n,n}(x)(1-x)^{\alpha}(1+x)^{\beta} x^{\gamma} dx = 0\,,\ k = 0, ..., n-1\,,$$

$$\int_{-1}^0 x^k P_{n,n}(x)(1-x)^{\alpha}(1+x)^{\beta} |x|^{\gamma} dx = 0\,,\ k = 0, ..., n-1\,,$$

where $\alpha, \beta, \gamma > -1$.

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

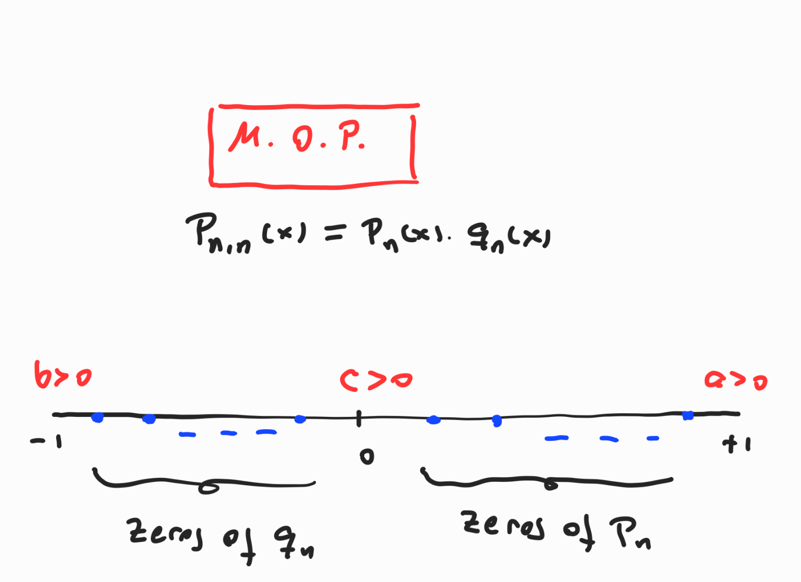
Since intervals $[-1,0]$ and $[0,1]$ have disjoint interiors $\Longrightarrow$ $P_{n,n}$ has exactly n simple zeros in each interval.

$$P_{n,n}(x) = p_n(x)\, q_n(x)\ ,$$

where $p_n(x) = \prod_{k=1}^{n} (x - x_k)$ and $q_n(x) = \prod_{k=1}^{n} (x - y_k)$, with $\{x_k\}_{k=1}^{n} \subset (0,1)$ and $\{y_k\}_{k=1}^{n} \subset (-1,0)$.

- Rodrigues Formula
- O.D.E. (3rd order!)

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?



Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

Jacobi case:

$$P_n^{(\alpha,\beta)} \text{ minimizes } \int_{-1}^{1} \Pi(x)^2 (1-x)^\alpha (1+x)^\beta \, dx$$

$$a = \frac{\alpha+1}{2}, \; b = \frac{\beta+1}{2}$$

Jacobi–Angelesco case:

$P_{nn}(x) = p_n(x) q_n(x)$, where

$$p_n \text{ minimzes } \int_{0}^{1} \Pi(x)^2 q_n(x) (1-x)^\alpha (1+x)^\beta \, x^\gamma \, dx$$

$$q_n \text{ minimzes } \int_{-1}^{0} \Pi(x)^2 p_n(x) (1-x)^\alpha (1+x)^\beta \, |x|^\gamma \, dx$$

$\Longrightarrow$ It suggest to try an electrostatic setting where the mutual repulsion between charges of the same interval is twice the corresponding repulsion between charges of different intervals!

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

$$E(x_1, \ldots, x_n; y_1, \ldots, y_n) =$$

$$2 \sum_{1 \le i < j \le n} \log\left(\frac{1}{|x_i - x_j|}\right) + 2 \sum_{1 \le i < j \le n} \log\left(\frac{1}{|y_i - y_j|}\right)$$

$$+ \sum_{i,j=1}^{n} \log\left(\frac{1}{|x_i - y_j|}\right) + a\left(\sum_{i=1}^{n} \log\left(\frac{1}{|x_i - 1|}\right) + \sum_{i=1}^{n} \log\left(\frac{1}{|y_i - 1|}\right)\right)$$

$$+ b\left(\sum_{i=1}^{n} \log\left(\frac{1}{|x_i + 1|}\right) + \sum_{i=1}^{n} \log\left(\frac{1}{|y_i + 1|}\right)\right)$$

$$+ c\left(\sum_{i=1}^{n} \log\left(\frac{1}{|x_i|}\right) + \sum_{i=1}^{n} \log\left(\frac{1}{|y_i|}\right)\right)$$

Do the zeros of $P_{n,n}$ minimize (or define a critical configuration, at least) of such energy functional? NOT!!!

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

ACTUAL SOLUTION:

A. Martínez Finkelshtein, R.O., J. Sánchez Lara (2021).

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

We associate to $P_{n,n}$ two electrostatic "partners" $S_{n,1}, S_{n,2}$, such that:

- $S_{n,1}, S_{n,2}$ have degree $n+1$
- At least $n-1$ zeros of $S_{n,1}$ (resp. $S_{n,2}$) lie on $(-1,0)$ (resp. $(0,1)$) and interlace with those of q_n (resp. p_n)

If we assign a charge of value $-1/2$ ("attractive") to each zero of $S_{n,1}$ (resp. $S_{n,2}$), then the zeros of $P_{n,n}$ define a critical configuration for each of the following electrostatic problems:

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

Equilibrium problem

- A charge of value $+1$ placed at each zero of $P_{n,n}$
- Positive charges of values $\frac{\alpha+1}{2}$, $\frac{\gamma+1}{2}$, $\frac{\beta+1}{2}$, placed, respect., at $x=1, x=0, x=-1$.
- Negative charges of values $-1/2$ placed at the zeros of $S_{n,1}$ (resp., $S_{n,2}$), $n-1$ of which interlace with those of q_n (resp., p_n).

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

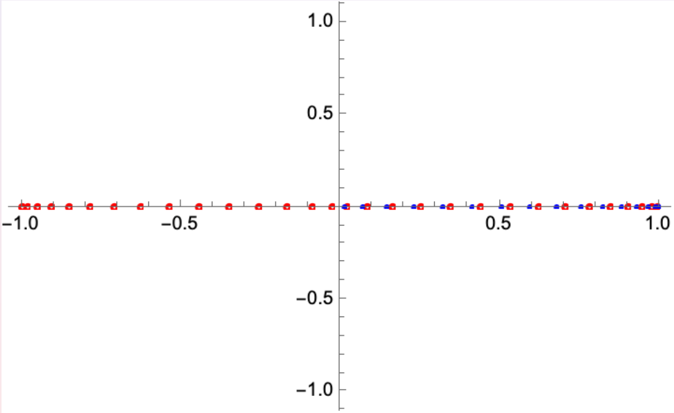


Figure: zeros of $p_{\mathbf{n}}$ (empty circles, all on $[-1,1]$) and of $S_{\mathbf{n},1}$ (filled circles, all on $[0,1]$) for $\mathbf{n} = (15,15)$.

Open problem: Hermite-Padé Polynomials (MOP). An electrostatic model?

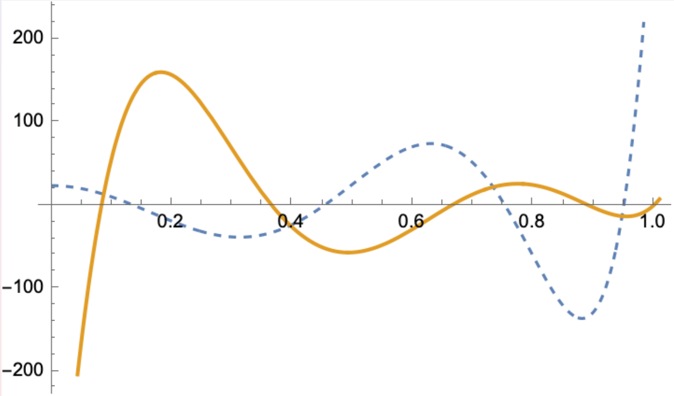


Figure: Graph of $p_{\mathbf{n}}$ (dashed line) and $S_{\mathbf{n},2}$ (thick line) on $[0,1]$ for $\mathbf{n} = (4,4)$ in the Appell's case: $\alpha = \beta = \gamma = 0$.

Multiple orthogonal polynomials of type II

- $\Delta_1, \Delta_2 \subset \mathbb{R}$, w_i positive weights on Δ_i , $i = 1, 2$, $\mathbf{n} = (n_1, n_2)$.
- $p_{\mathbf{n}}$, of degree $N = n_1 + n_2$, provided it exists, satisfying

$$\int_{\Delta_i} x^j p_{\mathbf{n}}(x) w_i(x) dx \begin{cases} = 0, & j \leq n_i - 1, \\ \neq 0, & j = n_i, \end{cases} \qquad i = 1, 2.$$

- We assume that both weights are semiclassical:
$\dfrac{w_i'(x)}{w_i(x)} = \dfrac{B_i}{A_i}$, $i = 1, 2$.
$\sigma_i := \max\{\deg(A_i) - 2, \deg(B_i) - 1\}, \quad i = 1, 2$.

Multiple orthogonal polynomials of type II

Now, $p_{\mathbf{n}}$ has two electrostatic partners: $S_{\mathbf{n},1}, S_{\mathbf{n},2}$, of respective degrees $n_2 + \sigma_1, n_1 + \sigma_2$.

Pair of ODEs

$p_{\mathbf{n}}$, of degree $N = n_1 + n_2$, assuming it exists, satisfies the pair of ODEs:

$$A_i S_{\mathbf{n},i}\, y'' + (A_i' S_{\mathbf{n},i} - A_i S_{\mathbf{n},i}' + B_i S_{\mathbf{n},i})\, y' + C_{\mathbf{n},i}\, y = 0.$$

Denoting by $x_{\mathbf{n},k}$, $k = 1, \ldots, N$, the zeros of $p_{\mathbf{n}}$,

$$y''(x_{\mathbf{n},k}) + \left(\frac{A_i'}{A_i} + \frac{B_i}{A_i} - \frac{S_{\mathbf{n},i}'}{S_{\mathbf{n},i}} \right)(x_{\mathbf{n},k}) y'(x_{\mathbf{n},k}) = 0$$

Multiple orthogonal polynomials of type II

Equilibrium problem

- A charge of value $+1$ placed at each zero of $p_{\mathbf{n}}$. These charges repel each other.
- Positive charges at the endpoints of the supporting intervals of the weights w_i , $i = 1, 2$. They are also repellent.
- Negative charges of values $-1/2$ placed at the zeros of $S_{n,1}$ (resp., $S_{n,2}$).

Multiple orthogonal polynomials of type II

This general electrostatic model is still formal: in general, the fact that the degree of $p_{\mathbf{n}}$ is maximal, as well as the simplicity and location of its zeros are not guaranteed ("normality") $\Longrightarrow$
We need to impose additional restrictions

Particular cases studied in the literature

- Angelesco setting: Intervals with disjoint interiors.
- Nikishin setting: Matching intervals, different weights related by a condition.
- Rakhmanov and others' setting: Overlapping intervals and weights related by a Nikishin-type condition.

Angelesco setting

- $\Delta_1, \Delta_2 \subset \mathbb{R}$, $\dot{\Delta}_1 \cap \dot{\Delta}_2 = \emptyset$
- w_1, w_2 semiclassical weights. Polynomial $p_{\mathbf{n}}$ satisfies n_1 orthogonality conditions on Δ_1, and n_2 on Δ_2, with $\mathbf{n} = (n_1, n_2)$ and $N = n_1 + n_2$.

Under these assumptions, $p_{\mathbf{n}}$ is normal (of degree N) and has exactly n_i simple zeros in $\dot{\Delta}_i$, $i = 1, 2$.

electrostatic partners

Polynomial $S_{\mathbf{n},1}$ (respect. $S_{\mathbf{n},2}$) has $n_2 - 1$ (respect. $n_1 - 1$) zeros, out of a total of $n_2 + \sigma_1$ (resp. $n_1 + \sigma_2$) interlacing with those of $p_{\mathbf{n}}$ on Δ_2 (respect. Δ_1).

Angelesco setting

Electrostatic model

The $N = n_1 + n_2$ zeros of $p_{\mathbf{n}}$, equipped with unit positive charges, are in equilibrium in the external field created by the orthogonality weights w_1, w_2 and

- charges of value $-1/2$ ("attractors") placed at the zeros of $S_{\mathbf{n},1}$; or
- charges of value $-1/2$ ("attractors") placed at the zeros of $S_{\mathbf{n},2}$.

Angelesco setting

Example. APPELL's polynomials:

$w_1(x) = w_2(x) \equiv 1\,;\ \Delta_1 = [-1, 0]\,,\ \Delta_2 = [0, 1].$

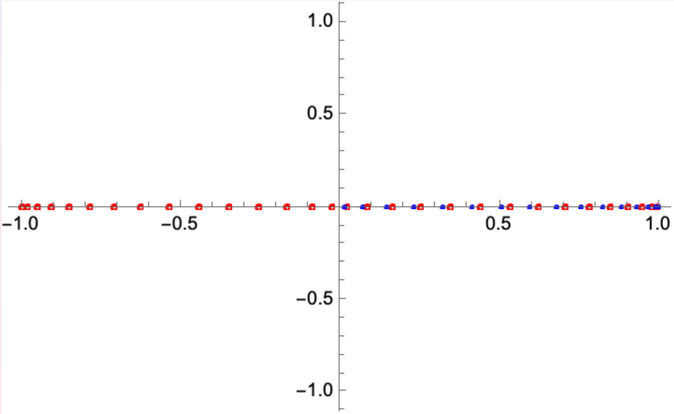


Figure: zeros of $p_{\mathbf{n}}$ (empty circles, all on $[-1, 1]$) and of $S_{\mathbf{n},1}$ (filled circles, all on $[0, 1]$) for $\mathbf{n} = (15, 15)$.

Nikishin setting

Nikishin (1980) proposed an elegant AT–system w_1, w_2. We present a slightly generalized version of it:

- $\Delta_1 = \Delta_2 = [a, b]$
- $\dfrac{w_2(x)}{w_1(x)} = |\Pi(x)|u(x)\,,\ u(x) = \int_c^d \frac{v(t)dt}{x-t}\,,\ (a, b) \cap (c, d) = \emptyset$
  and Π an arbitrary polynomial of degree m without zeros in $(a, b) \cup (c, d)$.

But we also have to impose an important restriction:

- w_1 and u must be semiclassical.

Example: $w_1(x) = |x - a|^\alpha\,|x - b|^\beta$,
$w_2(x) = |x - a|^\alpha\,|x - b|^\beta |x - c|^\gamma\,|x - d|^\delta$, $x \in (a, b)$, with $(a, b) \cap (c, d) = \emptyset$ and $\alpha, \beta, \gamma, \delta > -1.$ and $\gamma, \delta \notin \mathbb{Z}$, $\gamma + \delta \in \mathbb{Z}$.

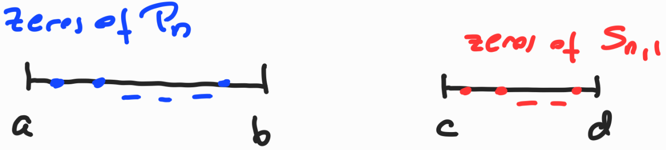
Nikishin setting

Electrostatic model

- $\ell = \min(n_2 - 1, n_1 - m), \quad m = \deg(\Pi)$
- $p_{\mathbf{n}}$ has at least $n_1 + \ell + 1$ sign changes on (a, b)
- $S_{\mathbf{n},1}$ has at least ℓ sign changes in (c, d).

If $n_2 \leq n_1 - m + 1$, so that $\ell = n_2 - 1 \implies$
$p_{\mathbf{n}}$ has exactly $N = n_1 + n_2$ simple zeros in (a, b), while $S_{\mathbf{n},1}$ has $\geq n_2 - 1$ zeros in (c, d), exactly as in the classical Nikishin setting $(m = 0)$.

NIKISHIN



$$\frac{w_2(x)}{w_1(x)} = |\pi(x)|\, u(x), \quad u(x) = \int_c^d \frac{d\sigma(t)}{x-t}$$

Rakhmanov's setting

Overlapping intervals: Aptekarev (2008), Aptekarev and Lysov (2011) ...

A particular Rakhmanov's case: Rakhmanov (2011)

- Diagonal setting: $n_1 = n_2 = n\,,\ \mathbf{n} = (n,n)\,,\ N = 2n$
- $\Delta_1 \subseteq \Delta_2$
- Nikishin type condition: $\dfrac{w_2(x)}{w_1(x)} = u(x)\,,$

  $u(x) = \int_{\Delta_3} \frac{v(t)dt}{x-t}\,,\ \dot{\Delta}_2 \cap \dot{\Delta}_3 = \emptyset$

Rakhmanov proved that at least $N - 5 = 2n - 5$ zeros of $p_{\mathbf{n}}$ lie on Δ_2.

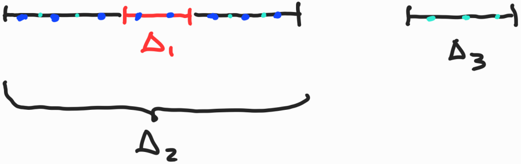
But ... what about its electrostatic partners $S_{\mathbf{n},i}$, $i = 1, 2$???

Observe that, by orthogonality, $p_{\mathbf{n}}$ has $n + r$ zeros in $\cdot\Delta_1$, with $0 \leq r \leq n$, and s zeros in $\Delta_2 \setminus \Delta_1$, with $0 \leq s \leq n$ and $r + s \leq n$. Then, we proved that:

Zeros of $S_{\mathbf{n},1}$

$S_{\mathbf{n},1}$ has at least $s - 2$ zeros in $\Delta_2 \setminus \Delta_1$, which interlace with those of $p_{\mathbf{n}}$ placed there, and at least $r - 3$ in Δ_3.

OVERLAPPING INTERVALS
(RAKHMANOV'S CASE)



- zeros of P_n
- zeros of $S_{n,1}$

Electrostatic partners

BUT...WHO THE HECK ARE THESE GUYS???

$S_{\mathbf{n},i}$, $i = 1, 2$???

Electrostatic partners

$$S_{\mathbf{n},i} := D_{w_i}[p_{\mathbf{n}}] = \det \begin{pmatrix} p_{\mathbf{n}} & \widehat{p}_{\mathbf{n},i} \\ A_i p'_{\mathbf{n}} & A_i \left(\widehat{p}_{\mathbf{n},i}\right)' - B_i \widehat{p}_{\mathbf{n},i} \end{pmatrix}$$

$$\frac{w'_i}{w_i} = \frac{B_i}{A_i}\,,\ \widehat{p}_{\mathbf{n},i}(x) = \int_{\Delta_i} \frac{p_{\mathbf{n}}(t) w_i(t)}{t - x}\, dt$$

Electrostatic partners

We have applied this electrostatic approach to several examples studied in the literature:

- Jacobi polynomials with non-standard values of parameters.
- Multiple Hermite polynomials.
- Multiple Laguerre polynomials of first and second kind.
- Jacobi-Piñeiro polynomials.
- Angelesco-Jacobi polynomials.
- Multiple orthogonal polynomials for the cubic weight.

THANK YOU SO MUCH!!!

MUCHAS GRACIAS!!!

Teide: Volcano in Tenerife (January 2022)