Sharp Balian-Low Theorems and Fourier Multipliers

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Department of Mathematics

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Joint work with: Shahaf Nitzan & Michael Northington

Notation

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- Translation: $T_s f(x) = f(x - s)$
- Modulation: $M_r f(x) = e^{2\pi i r x} f(x)$
- For fixed $a, b > 0$ and $f \in L^2(\mathbb{R})$ define:

$$f_{m,n}(x) = M_{mb} T_{an} f(x) = e^{2\pi i b m x} f(x - na)$$

- Gabor system: $G(f, a, b) = \{f_{m,n}\}_{m,n \in \mathbb{Z}}$

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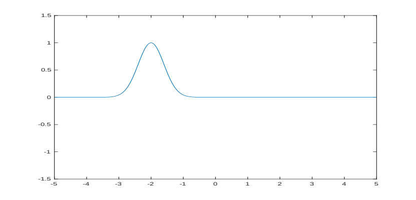

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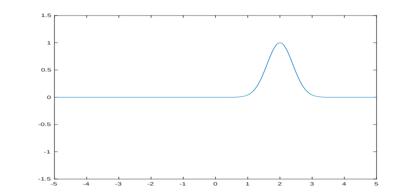

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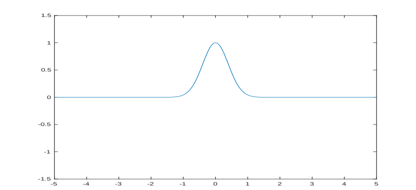

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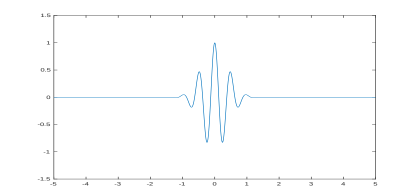

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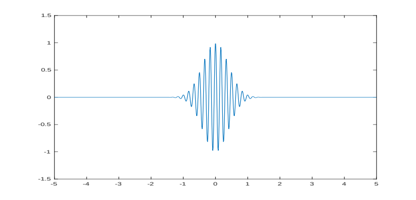

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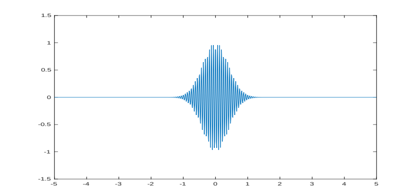

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Sometimes useful to visualize Gabor systems in the time-frequency plane:

- Recall Fourier transform: $\widehat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}dx$
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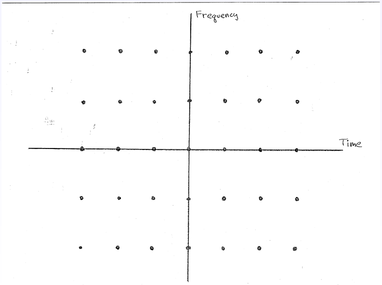

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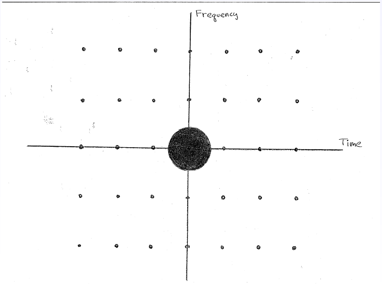

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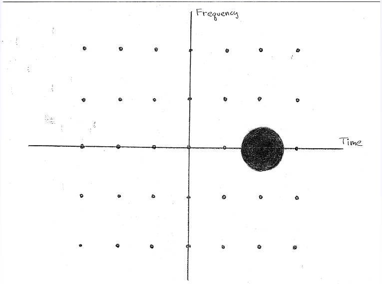

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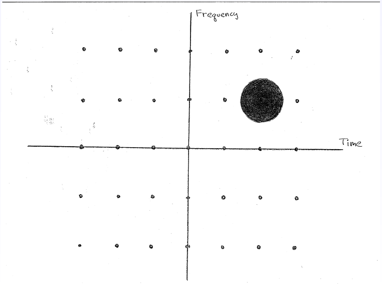

Signal representations

Goal: represent functions/signals h in terms of $G(f, a, b)$

$$h(x) = \sum_{m,n\in\mathbb{Z}} c_{m,n} f_{m,n}(x) = \sum_{m,n\in\mathbb{Z}} c_{m,n} e^{2\pi ibmx} f(x - an)$$

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- Gabor system applications: communications engineering (OFDM),
  audio signal processing, optics, physics,
  analysis of pseudodifferential operators,
  subfamily of Fefferman-Cordoba wavepackets

Example 1

General problem: How to chose $f \in L^2(\mathbb{R})$ and $a, b > 0$
so that $G(f, a, b)$ nicely spans $L^2(\mathbb{R})$?

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- Let $f(x) = \chi_{[0,1]}(x) =$ indicator function of $[0,1]$
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- f is poorly localized in frequency since $|\widehat{f}(\xi)| = \left|\frac{\sin(\pi\xi)}{\pi\xi}\right| \sim \frac{1}{|\xi|}$
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- Remark: Gabor ONB only occur on time-freq lattices of density one
  $G(h, a, b)$ is an ONB for $L^2(\mathbb{R}) \implies ab = 1$

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- Let $g(t) = e^{-\pi x^2}$
- Suppose $0 < ab < 1$ and consider $G(g, a, b) = \{g_{m,n}\}_{m,n \in \mathbb{Z}}$
- Then every $f \in L^2(\mathbb{R})$ has an unconditionally convergent Gabor expansion:

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However:

- The above $G(g, a, b)$ is not an ONB for $L^2(\mathbb{R})$ (but it's a "frame")
- $G(g, a, b)$ provides redundant (non-unique) representations
- $G(g, a, b)$ poorly conditioned when $ab \approx 1$

Well-localized Gabor ONB?

So far:

- $G(\chi_{[0,1]}, 1, 1)$ is ONB for $L^2(\mathbb{R})$; poor frequency localization
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- If $G(f, 1, 1)$ is an ONB for $L^2(\mathbb{R})$, can f, $\widehat{f}$ both be well-localized?
- Balian-Low theorem is a fundamental obstruction that prevents this...

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Balian-Low Theorem

Suppose $f \in L^2(\mathbb{R})$ satisfies

$$\int |x|^2 |f(x)|^2 dx < \infty \quad \text{and} \quad \int |\xi|^2 |\widehat{f}(\xi)|^2 d\xi < \infty.$$

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- BLT holds more generally for Riesz bases
- BLT is sharp (2003): fails with weights $\frac{|x|^2}{\log^{2+\epsilon}(|x|+2)}$, $\frac{|\xi|^2}{\log^{2+\epsilon}(|\xi|+2)}$
- $\exists$ versions with nonsymmetric weights $|x|^p$, $|\xi|^{p'}$, Gautam (2008)

Perspective

Heisenberg Uncertainty Principle

If $f \in L^2(\mathbb{R})$ and $\|f\|_{L^2(\mathbb{R})} = 1$ then:

$$\left(\int |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\int |\xi|^2 |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \geq 4\pi$$

- Heisenberg UP: f and $\widehat{f}$ cannot both be too well localized
- Balian-Low Theorem is a strong form of UP for Gabor ONB: $G(f,1,1)$ ONB for $L^2(\mathbb{R}) \implies$ lower bound in UP is infinite

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A weaker spanning structure than ONB:

- $\{f_n\} \subset L^2(\mathbb{R})$ is *minimal* if: $\quad \forall N, \; f_N \notin \overline{\mathrm{span}}\{f_n : n \neq N\}$
- $\{f_n\} \subset L^2(\mathbb{R})$ is *exact* if it is complete and minimal
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Exact BLT (Daubechies & Janssen, 1993)

Suppose $f \in L^2(\mathbb{R})$ satisfies

$$\int |x|^4 |f(x)|^2 dx < \infty \quad \text{and} \quad \int |\xi|^4 |\widehat{f}(\xi)|^2 d\xi < \infty.$$

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- $|x|^4, |\xi|^4$ weights are sharp
- $\exists$ versions with nonsymmetric weights $|x|^p, |\xi|^q$, (Heil & AP, 2007)

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Fix $2 \le q \le \infty$.
$\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is a (C_q)-system if every $f \in L^2(\mathbb{R})$ can be approximated arbitrarily well by finite sums $\sum a_n f_n$ with

$$\left(\sum |a_n|^q\right)^{1/q} \le C\|f\|_{L^2(\mathbb{R})}$$

- (C_q)-system provides completeness with ℓ^q control on coefficients
- Every ONB for $L^2(\mathbb{R})$ is a (C_2)-system
- If a Gabor system $G(f,1,1)$ is exact, then it is a (C_∞)-system

BLT for (C_q)-systems

Theorem (Nitzan, Northington, AP, 2017)

Fix $2 \leq q \leq \infty$. Suppose $f \in L^2(\mathbb{R})$ satisfies

$$\int |x|^{4(1-\frac{1}{q})} |f(x)|^2 dx < \infty \quad \text{and} \quad \int |\xi|^{4(1-\frac{1}{q})} |\widehat{f}(\xi)|^2 d\xi < \infty.$$

Then $G(f, 1, 1,)$ cannot be an exact (C_q)-system for $L^2(\mathbb{R})$.

BLT for (C_q)-systems

Theorem (Nitzan, Northington, AP, 2017)

Fix $2 \leq q \leq \infty$. Suppose $f \in L^2(\mathbb{R})$ satisfies

$$\int |x|^{4(1-\frac{1}{q})} |f(x)|^2 dx < \infty \quad \text{and} \quad \int |\xi|^{4(1-\frac{1}{q})} |\widehat{f}(\xi)|^2 d\xi < \infty.$$

Then $G(f,1,1,)$ cannot be an exact (C_q)-system for $L^2(\mathbb{R})$.

- The weights $|x|^{4(1-\frac{1}{q})}$, $|\xi|^{4(1-\frac{1}{q})}$ are sharp
- Almost sharp predecessor: Nitzan, Olsen, 2011
- $\exists$ sharp nonsymmetric versions of the theorem
- Note: $q = 2$ recovers BLT for ONB with weights $|x|^2, |\xi|^2$
  $q = \infty$ recovers BLT for exact systems with weights $|x|^4, |\xi|^4$

Ingredient 1 in the proof: the Zak transform

Zak transform

$$\forall (x,\xi) \in \mathbb{R}^2, \qquad Zf(x,\xi) = \sum_{k\in\mathbb{Z}} f(x-k)e^{-2\pi i k\xi}$$

- The Zak transform is quasiperiodic:

$$Zf(x,\xi+1) = Zf(x,\xi)$$
$$Zf(x+1,\xi) = e^{-2\pi i\xi} Zf(x,\xi)$$

  So $Zf(x,\xi)$ fully determined by values on square $(x,\xi) \in [0,1]^2$
- $Z : L^2(\mathbb{R}) \to L^2([0,1]^2)$ is unitary
- Zak transform converts Gabor system to windowed exponentials:

$$Z(f_{m,n}) = Z(M_m T_n f)(x,\xi) = e^{2\pi i(mx+n\xi)} Zf(x,\xi)$$

Zak transform

- Zak properties lead to isometric isomorphism between $L^2(\mathbb{R})$ and weighted space $L^2_W(\mathbb{T}^2)$ with weight $W(x,\xi) = |Zf(x,\xi)|^2$
- Weighted L^2 space:

$$\|F\|_{L^2_W(\mathbb{T}^2)} = \left(\int_0^1 \int_0^1 |F(x,\xi)|^2 W(x,\xi)\, dx\, d\xi \right)^{1/2}$$

- Consequence: spanning props of $G(f,1,1)$ in $L^2(\mathbb{R})$ correspond to spanning props of $\mathcal{E} = \{e^{2\pi i \langle k,z\rangle}\}_{k\in\mathbb{Z}^2}$ in $L^2_W(\mathbb{T}^2)$

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  - $G(f,1,1)$ is an exact (C_q)-system in $L^2(\mathbb{R}) \iff$???

Ingredient 2 in the proof: Fourier multipliers

Fix a function g. Fourier multiplier $\mathcal{T}_g$ is an operator that does:

$$\text{input} \longrightarrow \boxed{\mathcal{F}} \longrightarrow \boxed{\text{Multiply by } g} \longrightarrow \boxed{\mathcal{F}^{-1}} \longrightarrow \text{output}$$

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- Input is a sequence $c = \{c_k\}_{k\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$ and

$$\forall \xi \in \mathbb{T}^2, \quad \mathcal{F}[c](\xi) = \sum_{k\in\mathbb{Z}^2} c_k e^{2\pi i \langle k,\xi\rangle}$$

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- Output is the sequence: $\mathcal{T}_g(c) = \mathcal{F}^{-1}(g\mathcal{F}(c))$

Bounded Fourier multipliers $\mathcal{M}_p^q(\mathbb{T}^2)$

- Say that $g \in \mathcal{M}_p^q(\mathbb{T}^2)$ if $\mathcal{T}_g : \ell^p(\mathbb{Z}^2) \to \ell^q(\mathbb{Z}^2)$ is bounded operator

$$\|\mathcal{T}_g(c)\|_{\ell^q(\mathbb{Z}^2)} \leq K\|c\|_{\ell^p(\mathbb{Z}^2)}$$

- No nice characterization of $\mathcal{M}_p^q(\mathbb{T}^2)$, except for special cases

Lemma

Suppose $f \in L^2(\mathbb{R})$ and $2 \leq q \leq \infty$. Then:

$$G(f,1,1) \text{ is an exact } (C_q)\text{-system for } L^2(\mathbb{R}) \iff \frac{1}{|Zf|} \in \mathcal{M}_2^q(\mathbb{T}^2)$$

Ingredient 3 in the proof: an embedding for Zf

Lemma (Gautam, 2007)

Fix $2 \leq q \leq \infty$. Suppose $f \in L^2(\mathbb{R})$ satisfies

$$\int |x|^{4(1-\frac{1}{q})} |f(x)|^2 dx < \infty \quad \text{and} \quad \int |\xi|^{4(1-\frac{1}{q})} |\widehat{f}(\xi)|^2 d\xi < \infty.$$

If $s = 2(1 - 1/q)$ then for any $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} (|x|^2 + |\xi|^2)^{s/2} \, |\widehat{\varphi Zf}(x,\xi)|^2 dx d\xi < \infty.$$

- Lemma says that Zf embeds into Sobolev space $H^s_{loc}(\mathbb{R}^2)$
- Sobolev embeddings $\implies$ Zf has some regularity
  - when $q = 2$, $Zf \in VMO$
  - when $q > 2$, Zf has some Hölder continuity

Ingredient 4 in the proof: a topological fact

- Recall that the Zak transform $F = Zf$ is quasiperiodic:

$$F(x, \xi + 1) = F(x, \xi)$$
$$F(x + 1, \xi) = e^{-2\pi i \xi} F(x, \xi)$$

- Recall Sobolev embedding implies that $F = Zf$ is continuous

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Topological fact

A continuous quasiperiodic function must have a zero

Intuition:

- If x^* is fixed, then $F(x^*, \xi)$, $0 \leq \xi \leq 1$, is a closed curve $\Gamma_{x^*} \subset \mathbb{C}$
- $F(1, \xi) = e^{-2\pi i \xi} F(0, \xi)$
- Γ_0, Γ_1 have different winding numbers about 0

An obstruction to $\mathcal{M}_2^q(\mathbb{T}^d)$

Theorem (Nitzan, Northington, AP)

Suppose $2 \leq q \leq \infty$. The following three properties are incompatible:

1. $|F|^{-1} \in \mathcal{M}_2^q(\mathbb{T}^2)$
2. $F \in H^s_{loc}(\mathbb{R}^2)$, with $s = 2(1 - \frac{1}{q})$
3. F has a zero

- This implies the BLT for (C_q)-systems by taking $F = Zf$
- Extensions:
  - Nonsymmetric Sobolev spaces $H^{s_1,s_2}_{loc}(\mathbb{R}^2)$ ($\Longrightarrow$ nonsymmetric BLT)
  - Higher dimensions $\mathcal{M}_p^q(\mathbb{T}^d)$
  - Matrix-valued Fourier multipliers
  - Hausdorff dimension of zero set
  - Applications to multiply generated shift-invariant spaces

Summary

- Gabor systems $G(f, a, b)$ give time-freq representations of functions
- Balian-Low theorem: trade-off between $G(f, 1, 1)$ spanning structure
  and how well-localized that $f, \widehat{f}$ can be
- If $\int |x|^{4(1-\frac{1}{q})}|f(x)|^2 dx < \infty$ and $\int |\xi|^{4(1-\frac{1}{q})}|\widehat{f}(\xi)|^2 d\xi < \infty$,

  then $G(f, 1, 1,)$ cannot be an exact (C_q)-system for $L^2(\mathbb{R})$
- Unfortunately: only way for $G(f, a, b)$ to provide nicely concentrated time-frequency representations is to use overcomplete representations