Composition Operators on General Domains

Elodie Pozzi

Department of Mathematics and Statistics

Saint Louis University

joint work with B. R. Choe (Korea University), H. Koo (Korea University) and W. Smith (University of Hawaii)

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Composition Operators on Banach space

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$.

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For $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic. Then, the composition operator on $\mathcal{H}$ defined by

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There is a profusion of results on the composition operator when $\mathcal{H} = H^p$ where H^p is defined by

$$H^p := \left\{ f : \mathbb{D} \longrightarrow \mathbb{C} \text{ analytic } : \|f\|_{H^p} := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\},$$

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Question: What can be said about the composition operator on Hardy spaces on more general domains?

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Let $f \in H^p$. Then, f has an L^p-extension to $\partial\mathbb{D} = \mathbb{T}$ defined by

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Norm on H^p

Let $f \in H^p$. Then, $\operatorname{tr} f \in H^p(\mathbb{T}) := \{g \in L^p(\mathbb{T}) : \hat{g}(n) = 0,\ n < 0\}$ and

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<u>Extension to H^p</u>:

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Extension to H^p:

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t)g(e^{it})dt,\ P(r,\theta - t) = \frac{1-r^2}{1-2r\cos(\theta - t)+r^2},$$

with $g(e^{it}) = \operatorname{tr} f(e^{it})$ *a.e.*.

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$$\int_0^{2\pi} G(\phi(re^{i\theta}))d\theta \leq \int_0^{2\pi} G(re^{i\theta})d\theta.$$

Application: Littlewood Subordination Theorem + H^p definition *i.e.* via subharmonic majorant=boundedness of some composition operators C_ϕ with $\phi(0) = 0$.

Norm on H^2 and Area integral

Littlewood-Paley type Identity

Let $f \in H^2$. Then,

$$\frac{1}{2}\|f - f(0)\|_{H^2}^2 \leq \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z) \leq \|f - f(0)\|_{H^2}^2, \quad (1)$$

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In particular,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \simeq |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z). \tag{2}$$

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The Littlewood-Paley type Identity revisits the boundedness of C_ϕ for ϕ univalent:

$$\|C_\phi(f)\|_{H^2} \leq 3\|f\|_{H^2}.$$

Hardy spaces on domains Ω

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Domains such that $|\tau'(z)| < b$ have a rectifiable boundary in the sense:

$$\Lambda(\partial\Omega) = \frac{1}{2\pi} \int_0^{2\pi} |\tau'(e^{i\theta})| d\theta < \infty,$$

where Λ is the 1-Hausdorff measure restricted to the boundary $\partial\Omega$.

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Example for 1):

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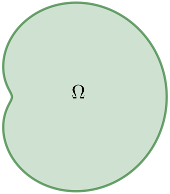

A Dini-smooth domain

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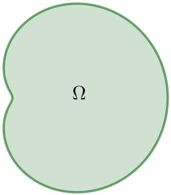

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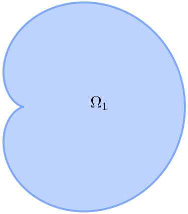
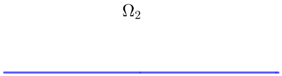
Example for 2):



Bounded domain with a cusp;



Unbounded domain: the upper half-plane

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Case of $a < |\tau'| < b$

The space of analytic functions $F : \Omega \longrightarrow \mathbb{C}$ such that

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Case of $|\tau'|$ not bounded above nor below

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where $\Gamma_r = \tau(r\mathbb{T})$ is the $E^p(\Omega)$ space and

$$F \in E^p(\Omega) \Longrightarrow (F \circ \tau) \cdot \tau' \in H^p.$$

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Rectifiable vs non rectifiable domain

A simply-connected domain Ω is rectifiable if $\Lambda(\partial\Omega) < \infty$ which is equivalent to

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Examples. • $\tau' \in H^1$: bounded chord-arc domains (*Lavrentiev domains*) which are domains for which the boundary $\partial\Omega$ satisifes

$$\Lambda(\partial\Omega(a, b)) \leq M|a - b|,$$

where $\partial\Omega(a, b)$ is the shorter arc of $\partial\Omega$ between a and b. Squares are bounded chord-arc domains.

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the strip $S = \{z \in \mathbb{C} : -1 < \mathrm{Im}(z) < 1\}$ is not a chord-arc domain.

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The existence of a trace to the boundary relies on the existence of a trace for τ'.

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Littlewood-Paley type Identity (Jerison-Kenig 1982)

Let $F \in E^2(\Omega)$. Then,

$$\int_{\partial\Omega} |f(z)|^2 |dz| \simeq \iint_{\Omega} |F'(w)|^2 \delta_{\Omega}(w) dA(w), \tag{3}$$

with constants depending only on the chord-arc constant.

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Assume that Ω is a **bounded** chord-arc domain with $0 \in \Omega$ and $\delta_{\partial\Omega}(0) \simeq \operatorname{diam}(\Omega) = 1$ where $\delta_\Omega(0)$ denotes the distance between 0 and $\partial\Omega$ and $\operatorname{diam}(\Omega)$ is the diameter.

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The same is valid if Ω is an unbounded chord-arc domain with any geodesic instead of the geodesic Γ_{z_0}.

$E^p(\Omega)$ spaces for Ω Carleson domain

A measure μ on Ω is a *Carleson measure* if

$$\|\mu\| := \sup_{\substack{z\in\partial\Omega \\ r>0}} \frac{1}{r}|\mu|(D(z,r)) < \infty,$$

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Carleson domain

A *Carleson domain* Ω is a simply-connected domain (bounded or unbounded) for which there exists $C > 0$ such that for $\mu \in \Delta(\Omega)$,

$$\iint_\Omega |f(z)||d\mu(z)| \leq C\|\mu\|\|f\|_1, \quad f \in E^1(\Omega), \tag{7}$$

(defined by Zinsmeister, *Les Domaines de Carleson*, 1985).

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Characterization of Carleson domains (Zinsmeister, 1985)

Ω is a Carleson domain if and only if $\log \tau' \in \mathrm{BMOA}(\mathbb{D})$ where $\mathrm{BMOA}(\mathbb{D})$ is the space of analytic functions b on $\mathbb{D}$ such that $b \in \mathrm{BMO}(\mathbb{T})$.

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The strip $S = \{z \in \mathbb{C} : -1 < \mathrm{Im}(z) < 1\}$ is a Carleson domain but is not a chord-arc domain.

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An example of a Carleson domain but not a chord-arc domain is a domain with a long cusp.

Littlewood-Paley type identity for the strip

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Let $F \in E^2(S)$. Then,

$$\int_{\partial S} |f(z)|^2 |dz| \simeq \int_{\mathbb{R}} |F(x)|^2 dx + \iint_S |F'(w)|^2 \delta_S(w) dA(w). \tag{8}$$

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Consequence

The Littlewood-Paley type identity can be obtained for Carleson domains that can be decomposed into a union of bounded chord-arc domains with the same chord-arc constant.

Reproducing kernel and Composition Operator

For $b \in \Omega$, the evaluation map $F \longmapsto F(b)$ is bounded on $E^2(\Omega)$ which gives the existence of $K_b \in E^2(\Omega)$ such that

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Let $g = \tau^{-1}$.

$$K_b(w) = \frac{(\overline{g'(b)}g'(w))^{1/2}}{1 - \overline{g(b)}g(w)} \qquad b, w \in \Omega.$$

with

$$\|K_b\|^2_{E^2(\Omega)} = \langle K_b, K_b \rangle = \frac{|g'(b)|}{1 - |g(b)|^2} \approx \frac{1}{\delta_\Omega(b)}.$$

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The reproducing kernel K_b behaves in the following way under C_ϕ^*: for $b \in \Omega$,

$$C_\phi^*(K_b) = K_{\phi(b)}.$$

Reproducing Kernel Thesis

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Let Ω be a Carleson domain and let ϕ be an analytic self-map of Ω. Then C_ϕ is bounded on $E^2(\Omega)$ if and only if

$$\sup_{b\in\Omega} \|K_b\|^{-2}_{E^2(\Omega)} \|C_\phi K_b\|^2_{E^2(\Omega)} < \infty. \tag{9}$$

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But C_ϕ^* does not necessarily satisfy the Reproducing Kernel Thesis on any Carleson domain.

Let $S = \{z \in \mathbb{C} : -1 < \mathrm{Im}(z) < 1\}$ and $\phi : S \longrightarrow S$ be defined as $\phi(z) = 0$. Then, C_ϕ^* is not bounded but

$$\sup_{b\in\Omega} \|K_b\|^{-2}_{E^2(\Omega)} \|C_\phi^* K_b\|^2_{E^2(\Omega)} < \infty. \tag{10}$$

Thank You!