COVERING RADIUS OF RANDOMLY DISTRIBUTED POINTS

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IN THE MEMORY OF V. P. HAVIN

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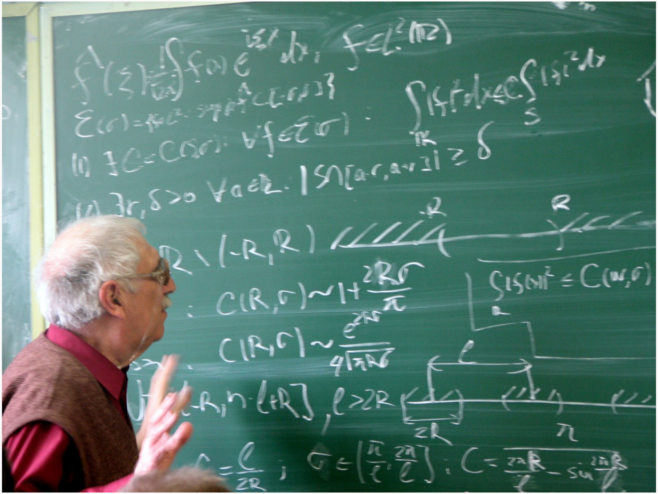

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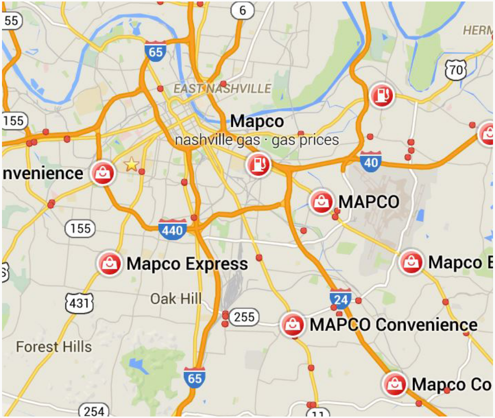

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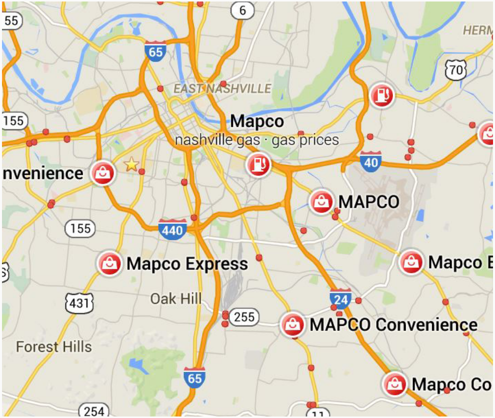

- There are two pumps very close to each other ☹
- However, they "cover" the map pretty good: every vehicle has a pump at a relatively small distance ☺

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SEVERAL PROPERTIES OF FINITE SETS

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If μ is a probability measure on $\mathcal{X}$, define
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Covering radius

Define $\rho(X_N) = \sup_y \min_i |y - x_i|$.

It measures the largest ball that does not intersect X_N.

Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, equipped with the normalized surface measure. The following holds:

- For any set X_N: $\delta(X_N) \leqslant \frac{c_1}{\sqrt{N}}$, and this bound is attained;

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- For any set X_N: $\rho(X_N) \geqslant \frac{c_2}{\sqrt{N}}$, and this bound is attained;
- There is an example of a set X_N for which

$$D(X_N) \leqslant \frac{c_3\sqrt{\log N}}{N^{3/4}}.$$

We briefly explain how to obtain sets X_N with almost perfect separation and covering properties. Suppose $s > 0$, and $X_N = \{x_1, \ldots, x_N\}$ is a set that minimizes

$$\min \sum_i \sum_{j \neq i} \frac{1}{|x_i - x_j|^s},$$

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- If $0 < s < 2$ or $s > 2$ then $\delta(X_N) \geqslant \frac{c_1}{\sqrt{N}}$ (Damelin, Dragnev, Kuijlaars, Maymeskul, Saff, Sun);
- If $s = 1$ or $s > 2$ then $\rho(X_N) \leqslant \frac{c_2}{\sqrt{N}}$ (Dahlberg, Damelin, Maymeskul).

We consider $\mathbb{S}^3$. For a positive integer n, denote

$$\mathcal{E}(n) := \frac{1}{\sqrt{n}}\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = n \colon x_1, x_2, x_3, x_4 \in \mathbb{Z}\}.$$

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In the same paper Bourgain, Rudnick and Sarnak show that

$$\mathbb{E}\rho(X_N) \leqslant N^{-1/3+\varepsilon},$$

where $X_N = \{x_1, \ldots, x_N\}$ are uniformly distributed over $\mathbb{S}^3$.

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The proof relies on the following uniform limit:

$$r^{-s} m_s(B(x,r) \cap S) \rightrightarrows \omega_s, \; r \to 0, \quad x \in S.$$

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Example

The above limit is clearly uniform for a sphere $\mathbb{S}^d$. Thus, the constant we get is

$$\left(\frac{(d+1)\omega_{d+1}}{\omega_d}\right)^{1/d}.$$

To achieve the constant $\left(\frac{m_s(S)}{\omega_s}\right)^{1/s}$, it was essential (and sufficient!) to have $r^{-s}m_s(B(x,r)\cap S) \rightrightarrows \omega_s$ as $r \to 0$.

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More natural examples of such a situation is a unit ball $B(0,1) \subset \mathbb{R}^d$ and a unit cube $[0,1]^d \subset \mathbb{R}^d$. While at almost every point x it holds that $r^{-d} m_d(B(x,r) \cap K) \to \omega_d$, it is not uniform, and on the boundary the limit is not equal to ω_d.

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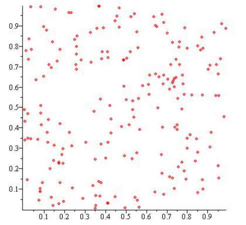
Theorem

$$\lim_{N\to\infty} \mathbb{E}\rho(X_N; B(0,1)) \cdot \left[\frac{N}{\log N}\right]^{1/d} = \left(\frac{2(d-1)}{d}\right)^{1/d} \quad \text{compare to } 1;$$

$$\lim_{N\to\infty} \mathbb{E}\rho(X_N; [0,1]^d) \cdot \left[\frac{N}{\log N}\right]^{1/d} = \left(\frac{2^{d-1}}{d\omega_d}\right)^{1/d} \quad \text{compare to } (1/\omega_d)^{1/d}.$$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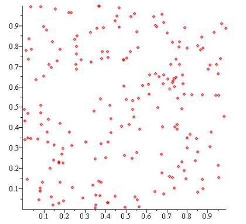
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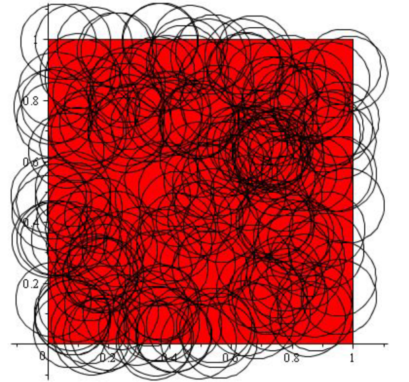
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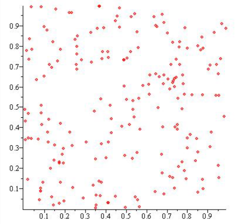
PICTURES



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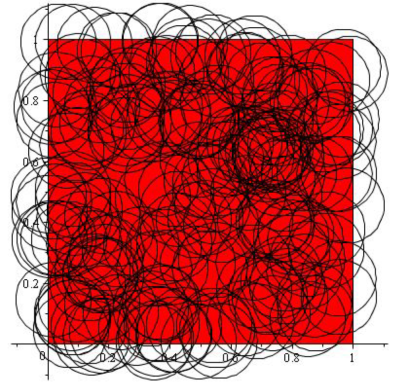

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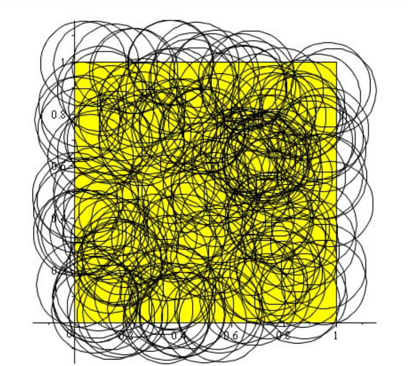

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We can handle this case with a two-sided estimate instead of the asymptotic behavior. That is,

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Suppose $\mathcal{X}$ is a compact metric space, equipped with a probability Borel measure μ. We assume that there exists a positive, strictly increasing function $\Phi\colon \mathbb{R} \to \mathbb{R}$ with following properties:

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- $$\mathbb{E}\rho(X_N) \asymp \Phi^{-1}\left(\frac{\log N}{N}\right).$$

APPLICATION: ε-COVERINGS

A set X_N is an ε-covering of $\mathcal{X}$, if any point of $\mathcal{X}$ can be approximated by a point of X_N up to ε. On a smooth s-dimensional manifold the best ε-covering will have cardinality $\frac{c}{\varepsilon^s}$.

APPLICATION: ε-COVERINGS

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Random ε-covering

Suppose $N \approx \varepsilon^{-s} \log(1/\varepsilon)$. Then, with high probability, a random set X_N of N points is an ε-covering of a smooth s-dimensional manifold K.

Thank you!