Fourier restriction and well approximable numbers

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The restriction estimate (extension estimate)

Let μ be a positive Borel measure supported on a compact set in $\mathbb{R}^d$. We consider the estimate

$$\left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim_{p,q} \|f\|_{L^q(\mu)}$$

where

$$\widehat{fd\mu}(\xi) := \int f(x) e^{2\pi i x \xi} d\mu(x).$$

$A \lesssim_{p,q} B$ means $A \leq C_{p,q} B$ where the constant $C_{p,q}$ only depends on p and q.

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- If μ is a surface measure on a smooth manifold (for example, paraboloid, cone or moment curve), the estimate is related to dispersive PDEs.
- In this talk, we are interested in when μ is supported on a fractal set (for example, Cantor set).

The restriction estimate (extension estimate)

- We are interested when $q = 2$.

$$\left\| \widehat{f d\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \tag{1}$$

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- There is but two serious function space, and they are L^2 and L^1 (or L^2 and L^∞).

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- There is but two serious function space, and they are L^2 and L^1 (or L^2 and L^∞).
- We have trivial estimate

$$\left\| \widehat{fd\mu} \right\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mu)}$$

and we can interpolate with (1)

The restriction estimate (extension estimate)

Stein-Tomas theorem

Let μ be a surface measure on the $d-1$ dimensional paraboloid $\mathbb{P}^{d-1} := \{(x', |x'|^2) : x' \in [0,1]^{d-1}\}$. For $p \geq \frac{2(d+1)}{d-1}$,

$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)}$$

and the range of p is optimal.

The fact that the paraboloid has positive Gaussian curvature played a key role in the proof.

The restriction estimate

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- If μ is a surface measure on a smooth manifold (for example, paraboloid, cone or moment curve), we can use its geometric properties like dimension, smoothness and curvature.

The restriction estimate

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- If μ is a surface measure on a smooth manifold (for example, paraboloid, cone or moment curve), we can use its geometric properties like dimension, smoothness and curvature.
- However, if μ is supported on a fractal set (for example, Cantor set), we cannot use such geometric properties.

The restriction estimate

Theorem (Mockenhaupt, 2000, Mitsis, 2002, Bak-Seeger, 2011)

Let μ be a positive Borel measure on $\mathbb{R}^d$. Assume that there exists $a, b \in (0, d)$ such that

$$\mu(B(x,r)) \lesssim r^a \qquad \forall x \in \mathbb{R}^d, r > 0$$

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-b/2} \qquad \forall \xi \in \mathbb{R}^d.$$

For $p \geq (4d - 4a + 2b)/b$,

$$\left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)}.$$

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For $p \geq (4d - 4a + 2b)/b$,

$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)}.$$

- If μ is a surface measure on $\mathbb{P}^{d-1}$, then $a = b = d - 1$. Thus, $p \geq 2(d+1)/(d-1)$.
- Is the range of p optimal?

Optimality of the restriction theorem

For fixed a and b, there could exist many measures μ which satisfy

$$\text{Regularity}: \qquad \mu(B(x,r)) \lesssim r^a \qquad \forall x \in \mathbb{R}^d, r > 0$$

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For each $p < (4d - 4a + 2b)/b$, we want to construct **a measure** μ such that μ satisfies the regularity and Fourier decay, but

$$\left\| \widehat{f\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)}$$

fails.

Optimality of the restriction theorem

- Łaba and Hambrook (2013) and Chen (2016) : $0 \leq b \leq a < d = 1$.
- Łaba and Hambrook (2016): $d - 1 \leq b \leq a < d$.
- All these results are based on probabilistic constructions.

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- Our construction is deterministic.
- Actually, showed the optimality for all possible a, b when $d = 1$, since $b > 2a$ cannot happen. It was proved by Mitsis (2002).

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Theorem (Fraser,Hambrook and R., 2023+)

The range of p in the Mockenhaupt-Mitsis-Bak-Seeger restrictioin theorem is optimal if

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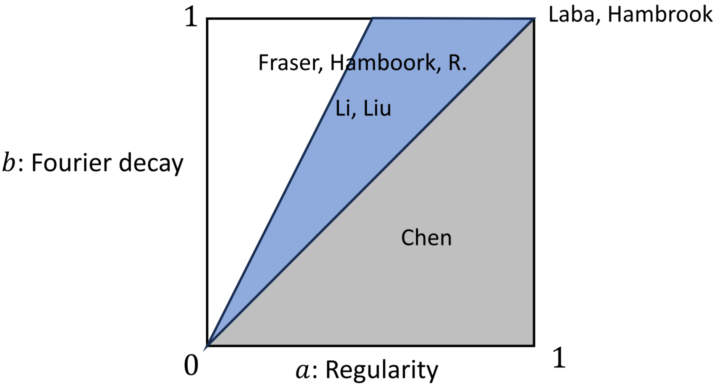
- Li and Liu (2024+) : another deterministic construction with other additional properties


1
Laba, Hambrook
Fraser, Hamboork, R.
Li, Liu
b: Fourier decay
Chen
0
a: Regularity
1

Beyond the range

For all measures, is the range of p optimal? No

$$\text{Regularity :} \qquad \mu(B(x,r)) \lesssim r^a \qquad \forall x \in \mathbb{R}^d, r > 0$$

$$\text{Fourier Decay :} \qquad |\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-b/2} \qquad \forall \xi \in \mathbb{R}^d$$

We want to construct a measure which satisfies the regularity and Fourier decay, but

$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$$

holds even when $p < (4d - 4a + 2b)/b$.

Beyond the range

Is the range of p optimal for all measures?
If μ is a measure supported on a set of Hausdorff dimension $\alpha < d$,

$$\left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)} \qquad \text{only when} \quad p \geq \frac{2d}{\alpha}.$$

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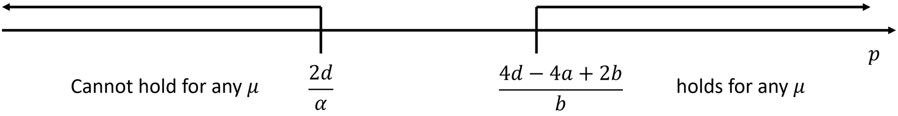
$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)} \qquad \text{only} \quad \text{when} \quad p \geq \frac{2d}{\alpha}.$$

It is well known in geometric measure theory that

$$0 \leq a, b \leq \alpha.$$

If $\alpha < d$, we have the following.



Beyond the range

Is there a measure μ supported on a set of Hausdorff dimension α such that

$$\left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \qquad \text{when } p = \frac{2d}{\alpha}? \tag{2}$$

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$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \qquad \text{when } p = \frac{2d}{\alpha}? \tag{2}$$

- Chen and Seeger(2017) : $\alpha = d/k$ and k is an integer.
- Shmerkin and Suomala(2017) : $d = 1$ and $0 < \alpha < 1/2$.
- Łaba and Wang(2018) : All α and d in nearly-optimal sense. In other words, (2) holds when $p > 2d/\alpha$.
- There is no explicit example known yet.

Fractal measure on a manifold

What if a fractal measure μ is supported on a smooth manifold?

- The paraboloid has positive curvature at any point. It leads to

$$\left\| \widehat{fd\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \tag{3}$$

  for $p \geq 2(d+1)/(d-1)$.
- How will the curvature affect to fractal sets?

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  for $p \geq 2(d+1)/(d-1)$.
- How will the curvature affect to fractal sets?
- Can we construct a measure μ **supported on the paraboloid** such that (3) holds even when $p < (4d - 4a + 2b)/b$?
  If it is possible, how small can p be?

Fractal measure on a manifold

What if a fractal measure μ is supported on a smooth manifold?

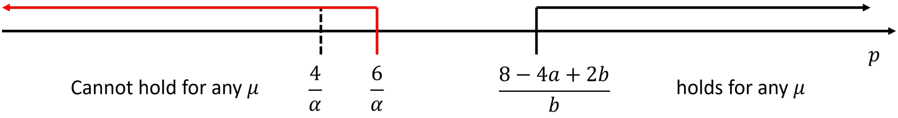
- (R. 2023) For $0 < \alpha < 1$, there exists a measure ν supported on **the parabola** $\mathbb{P} := \{(x, x^2) : x \in [0, 1]\}$ which satisfies the following:

What if a fractal measure μ is supported on a smooth manifold?

- (R. 2023) For $0 < \alpha < 1$, there exists a measure ν supported on **the parabola** $\mathbb{P} := \{(x, x^2) : x \in [0,1]\}$ which satisfies the following:
- $\dim_H(\mathrm{supp}(\nu)) = \alpha$
- For any $p > 6/\alpha$, we have

$$\left\| \widehat{f d\nu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\nu)}.$$

If $p < 6/\alpha$, the estimate above fails.



Summary

Let μ be a measure supported on a set of Hausdorff dimension α.

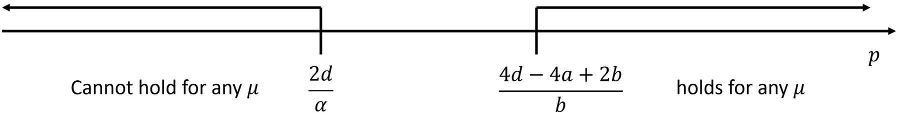
$$\mu(B(x,r)) \lesssim r^a \qquad \forall x \in \mathbb{R}^d, r > 0$$

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-b/2} \qquad \forall \xi \in \mathbb{R}^d.$$

Let us define the critical exponent of μ by

$$p_c(\mu) = \inf\{p : \left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \text{ holds}\}.$$

$$\frac{2d}{\alpha} \leq p_c(\mu) \leq \frac{4d-4a+2b}{b}$$



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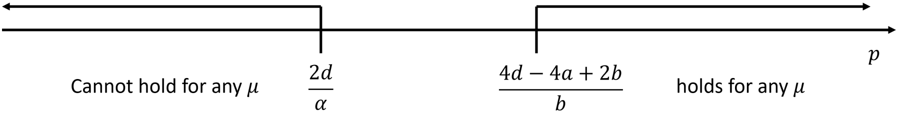

We have examples such that

- $p_c(\mu) = \frac{4d-4a+2b}{b}$
- $p_c(\mu) = \frac{2d}{\alpha}$
- $\frac{2d}{\alpha} < p_c(\mu) < \frac{4d-4a+2b}{b}$

Hausdorff dimension, regularity and Fourier decay are **not enough** to determine $p_c(\mu)$.

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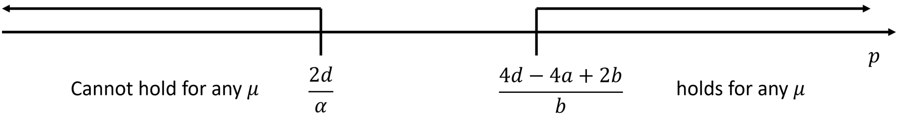

- Hausdorff dimension, regularity and Fourier decay are **not enough** to determine $p_c(\mu)$.
- Q: What are the determining factors of the optimal range of p?
- If we have explicit examples, we can hope to see a pattern.

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Result

Theorem (Fraser,Hambrook and R., 2023+)

The range of p in the Mockenhaupt-Mitsis-Bak-Seeger restrictioin theorem is optimal if

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For each $p < (4d - 4a + 2b)/b$, we want to construct **a measure** μ such that μ satisfies

$$\text{Regularity :} \qquad \mu(B(x,r)) \lesssim r^a \qquad \forall x \in \mathbb{R}^d, r > 0$$

$$\text{Fourier Decay :} \qquad |\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-b/2} \qquad \forall \xi \in \mathbb{R}^d,$$

but does not satisfies

$$\left\| \widehat{f\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)}.$$

Result

Theorem (Fraser,Hambrook and R., 2023+)

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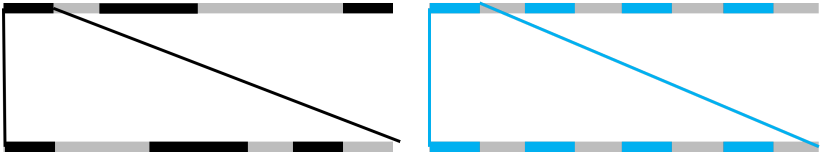
Main tools

- Well approximable numbers
- A cantor set determined by arithmetic progressions

Idea of the proof

Laba and Hambrook (2013) and Chen (2016) used two types of Cantor sets:
A random Cantor set C_R and **a Cantor set** C_D **determined by arithmetic progressions**.



Idea of the proof

- First, they constructed a measure μ on C_R such that, for any $\epsilon > 0$,

$$\mu(I) \lesssim_\epsilon |I|^{a-\epsilon} \qquad |\widehat{\mu}(\xi)| \lesssim_\epsilon (1+|\xi|)^{-a/2+\epsilon}$$

  where I is an interval in $\mathbb{R}$.

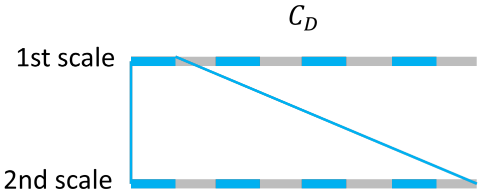
- They modified this measure so that the new measure μ is supported on $C_R \cup C_D$ and

$$\mu(I) \lesssim_\epsilon |I|^{a-\epsilon} \qquad |\widehat{\mu}(\xi)| \lesssim_\epsilon (1+|\xi|)^{-b/2+\epsilon}$$

  for $b \le a$.

Idea of the proof

Consider a sequence functions $\{f_k\}_{k\in\mathbb{N}}$ supported on C_D at the kth-scale.



$$\lim_{k\to\infty} \frac{\left\| \widehat{f_k d\mu} \right\|_{L^p(\mathbb{R}^d)}}{\|f_k\|_{L^2(\mu)}} = \infty, \qquad \text{if } p < \frac{4d-4a+2b}{b}$$

Thus, the range of p is optimal.

Idea of the proof

- **A random Cantor set** C_R and **a Cantor set** C_D **determined by arithmetic progressions**.

  $$\downarrow$$

  The set of α-well-approximable numbers $E(\alpha)$ and **a Cantor set** $C(\beta)$ **determined by arithmetic progressions**.

Idea of the proof

- **A random Cantor set** C_R and **a Cantor set** C_D **determined by arithmetic progressions**.

$$\downarrow$$

The set of α-well-approximable numbers $E(\alpha)$ and **a Cantor set** $C(\beta)$ **determined by arithmetic progressions**.

- To extend the range from $b \leq a$ to $b \leq 2a$, we gave **a different weight on** $C(\beta)$ according to the parameter β.

Set of well-approximable numbers

For $\alpha > 0$, we define the **set of α-well-approximable numbers** by

$$E(\alpha) := \{x \in \mathbb{R} : |x - r/q| \leq |q|^{-(2+\alpha)} \text{ for infinitely many } (q,r) \in \mathbb{Z}^2\}$$

The set $E(\alpha)$ arise from number theory; Major arc in circle method.

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- Kaufman (1981) proved that $E(\alpha)$ is a $\frac{2}{2+\alpha}$-dimensional **Salem set**. There exists a measure μ on $E(\alpha)$ such that

$$\dim_H(\text{supp}(\mu)) = \frac{2}{2+\alpha} \qquad \text{and} \qquad |\widehat{\mu}(\xi)| \lesssim_\epsilon (1+|\xi|)^{-\frac{1}{2+\alpha}+\epsilon}$$

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- Being a Salem set does not guarantee that $\mu(I) \lesssim_\epsilon |I|^{\frac{2}{2+\alpha}-\epsilon}$. But we constructed a measure which also satisfies the regularity.

Regularity and Fourier decay

- If $\beta = 0$, for any $\epsilon > 0$,

$$\mu(I) \lesssim_\epsilon |I|^{\frac{2}{2+\alpha}-\epsilon} \qquad |\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\frac{1}{2+\alpha}+\epsilon}$$

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- If $0 < \beta < 1$, the length of **the arithmetic sequence is long**, so that it lowers the exponent in the Fourier decay.

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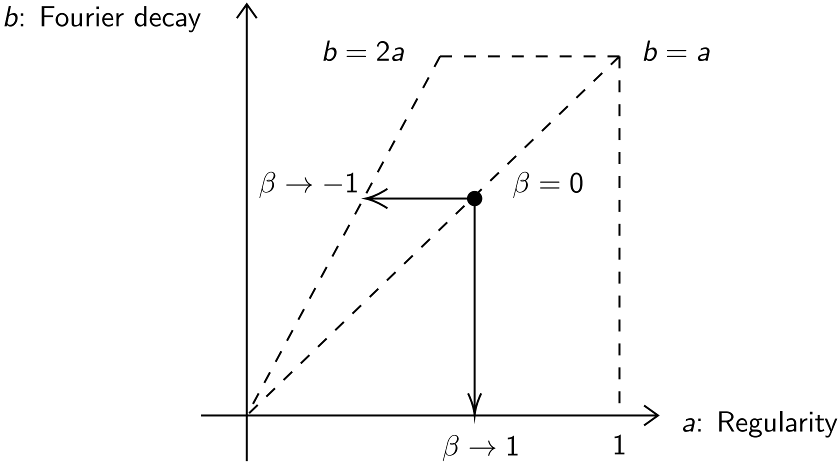
- If $-1 < \beta < 0$, **the weight on the Cantor set** $C(\beta)$ **is large**, so that it lowers the exponent in the regularity.

$$\mu(I) \lesssim_\epsilon |I|^{\frac{2+\beta}{2+\alpha}-\epsilon} \qquad |\widehat{\mu}(\xi)| \lesssim_\epsilon (1+|\xi|)^{-\frac{1}{2+\alpha}+\epsilon}$$

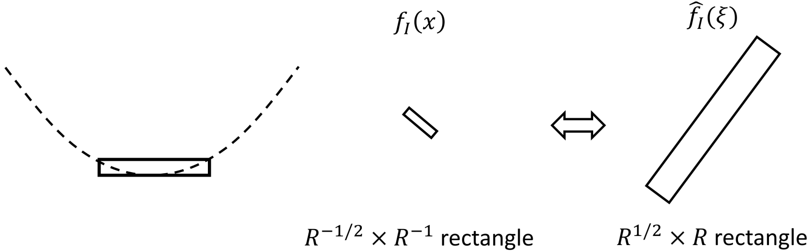


Figure

If we ignore ϵ in the exponents, $(a, b) = (\frac{2}{2+\alpha}, \frac{2}{2+\alpha})$ when $\beta = 0$.
$(a, b) \to (\frac{2}{2+\alpha}, 0)$ as $\beta \to 1$ and $(a, b) \to (\frac{1}{2+\alpha}, \frac{2}{2+\alpha})$ as $\beta \to -1$.

Knapp's example

Knapp's example gives an example such that L^2-L^p estimate fails when μ is supported on a smooth manifold.



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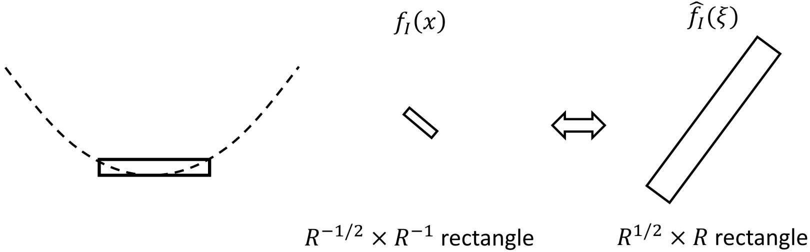

- $\left\| \widehat{f_I d\mu} \right\|_{L^p(\mathbb{R}^2)} \gtrsim R^{-1/2+3/2p}$ and $\|f_I\|_{L^2(\mu)} \approx R^{-1/4}$.
- $$\left\| \widehat{f d\mu} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \qquad \text{fails when} \quad p < 6.$$

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- In Knapp's example, we consider only one component.
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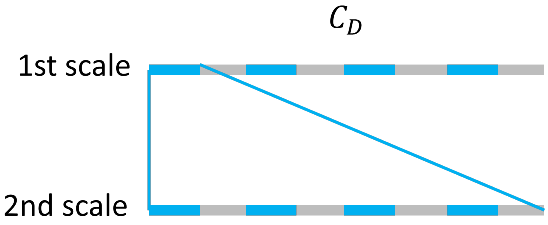
- In Knapp's example, we consider only one component.
- In fractal sets, we consider several components ordered in a certain way.
- $C(\beta)$: Cantor set determined by arithmectic progressions.
  We consider a smooth function f_k supported $C(\beta)$ at the kth scale.



Failure of the L^2-L^p estimate

We consider a smooth function f_k supported $C(\beta)$ at the kth scale.

$$\lim_{k\to\infty} \frac{\left\|\widehat{f_k d\mu}\right\|_{L^p(\mathbb{R}^d)}}{\|f_k\|_{L^2(\mu)}} = \infty \quad \text{whenever} \begin{cases} p < 2\frac{(1+\alpha-\beta)}{1-\beta} & \text{if } 0 \leq \beta < 1 \\ p < 2(1+\alpha-\beta) & \text{if } -1 < \beta < 0 \end{cases}$$

Failure of the L^2-L^p estimate

We consider a smooth function f_k supported $C(\beta)$ at the kth scale.

$$\lim_{k\to\infty} \frac{\left\|\widehat{f_k d\mu}\right\|_{L^p(\mathbb{R}^d)}}{\|f_k\|_{L^2(\mu)}} = \infty \quad \text{whenever} \begin{cases} p < 2\frac{(1+\alpha-\beta)}{1-\beta} & \text{if } 0 \leq \beta < 1 \\ p < 2(1+\alpha-\beta) & \text{if } -1 < \beta < 0 \end{cases}$$

- We have

$$\mu(I) \lesssim_\epsilon |I|^{\frac{2}{2+\alpha}-\epsilon} \text{ and } |\widehat{\mu}(\xi)| \lesssim_\epsilon (1+|\xi|)^{-\frac{1-\beta}{2+\alpha}+\epsilon} \qquad \text{if } 0 < \beta < 1$$

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- a : exponent in the regularity / b : exponent in the Fourier decay

$$\left\|\widehat{fd\mu}\right\|_{L^p(\mathbb{R}^d)} \lesssim_p \|f\|_{L^2(\mu)} \qquad \text{fails when } \; p < \frac{4d-4a+2b}{b}.$$

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Future direction

Theorem (Fraser,Hambrook and R., 2023+)

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- **Explicit Salem sets in higher dimensions**
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Algebraic Number Theory

For $\alpha > 0$, we define the **set of α-well-approximable numbers** by

$$E(\alpha) := \{x \in \mathbb{R} : |x - r/q| \leq |q|^{-(2+\alpha)} \text{ for infinitely many } (q, r) \in \mathbb{Z}^2\}$$

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Let $|\cdot|$ be the Euclidean norm. For $\tau > 1$,

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$$\frac{(1,1)}{(1,-1)} \simeq \frac{1+i}{1-i} = 0 + i \simeq (0,1)$$

Algebraic Number Theory

- K: a d dimensional field extension of $\mathbb{Q}$.
- $\mathbb{Z}_k$: a ring of integers of K.
- $B = \{w_1, \cdots, w_d\}$: an integral basis for K.
- $\mathbb{Z}_k \simeq \mathbb{Z}^d \quad K \simeq \mathbb{Q}^d \quad (q_1, \cdots, q_d) \simeq q_1 w_1 + \cdots + q_d w_d$
- $$\frac{r}{q} \simeq \frac{r_1 w_1 + \cdots + r_d w_d}{q_1 w_1 + \cdots + q_d w_d}$$

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  .
- If $K = \mathbb{Q}[i] := \{q_1 + q_2 i : q_1, q_2 \in \mathbb{Q}\}$, then $\mathbb{Z}_k = \mathbb{Z}[i], B = \{1, i\}$.

$$\frac{(1,1)}{(1,-1)} \simeq \frac{1+i}{1-i} = 0 + i \simeq (0,1)$$

Salem sets in higher dimension

Let $|\cdot|$ be the Euclidean norm.

$$E(K,B,\tau)$$
$$= \{x \in \mathbb{R}^d : \left|x - \frac{r}{q}\right| \le |q|^{-(\tau+1)} \text{ for infinitely many } (q,r) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$$

- For $\tau > 1$, they constructed a measure μ on $E(K,B,\tau)$ such that

$$\dim_H(\mathrm{supp}(\mu)) \le \frac{2d}{1+\tau} \qquad |\widehat{\mu}(\xi)| \lesssim_\epsilon |\xi|^{-\frac{d}{1+\tau}+\epsilon}$$

- **Can we repeat a similar argument in this setting?**

Questions

Thank you!