Random Matrices
and
Zeros of Polynomials

Guilherme Silva

Joint work with Pavel Bleher (IUPUI) [Memoirs of the AMS, to appear]

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size $N \times N$

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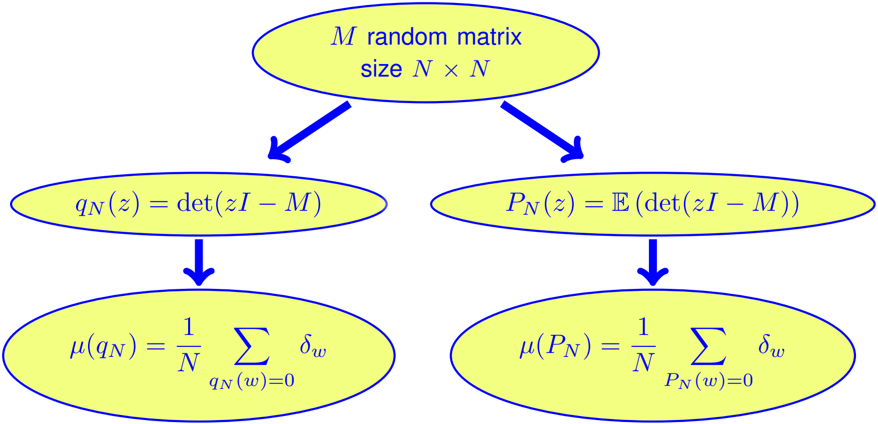

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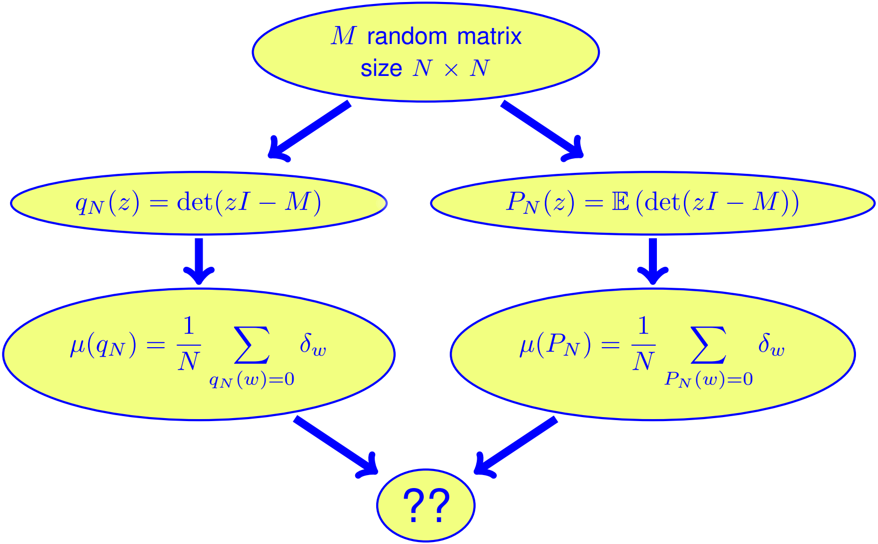
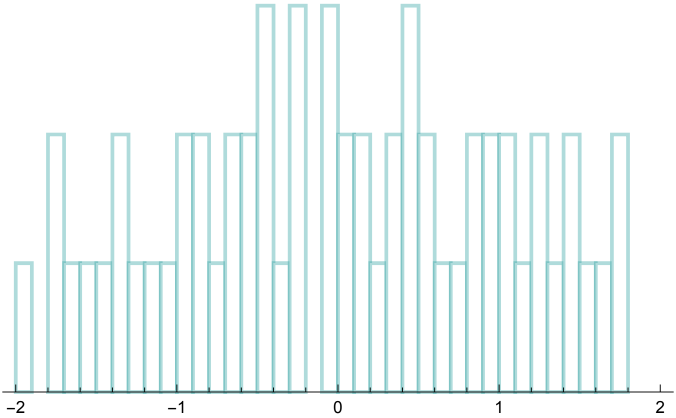


Eigenvalues for $N = 50, 60, \ldots, 1250$



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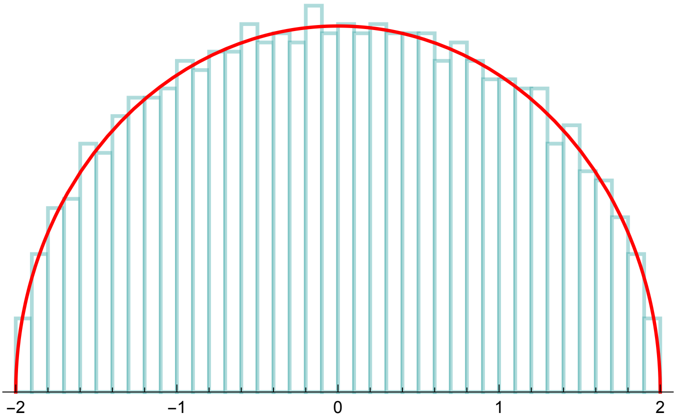

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- For D and Q nice enough, μ_Q uniquely exists
- If D is unbounded, we have to impose sufficient growth for Q
- Finding $\operatorname{supp} \mu_Q$ is challenging

Unitary ensembles

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- Unitary ensembles: space $\mathcal{H}_N$ of $N \times N$ hermitian matrices equipped with probability distribution

$$\frac{1}{\mathcal{Z}_N} e^{-N \operatorname{Tr} V(M)} dM, \tag{1}$$

where V is a real polynomial of even degree and dM is the Lebesgue measure on $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$.

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  - The factor N makes sure that eigenvalues remain bounded

Unitary ensembles - techniques

- We can see the diagonalization

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where dU is the Haar measure on $\mathcal{U}_N$
- Consequences:
  - Eigenvalues and eigenvectors are independent
  - Eigenvectors are uniformly distributed on $\mathcal{U}(N)$
  - Eigenvalues exhibit local repulsion

Unitary ensembles - global behavior of eigenvalues

We can rewrite

$$\frac{1}{Z_N}\prod_{j<k}(\lambda_j-\lambda_k)^2\prod_j e^{-NV(\lambda_j)}=\frac{1}{Z_N}e^{-N^2H(\lambda_1,\dots,\lambda_N)}$$

where

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- Thus the most likely eigenvalue configurations $\mu(q_N)$'s should be close to μ_V!

Unitary ensembles and orthogonal polynomials

After some massage, we get that

$$\frac{1}{Z_N}\prod_{j<k}(\lambda_j-\lambda_k)^2\prod_j e^{-NV(\lambda_j)}=\det(K_N(\lambda_k,\lambda_j))_{1\leq k,j\leq n}$$

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where K_N is the correlation kernel

$$K_N(x,y)=e^{-\frac{n}{2}(V(x)+V(y))}\sum_{k=0}^{N-1}p_k(x)p_k(y),$$

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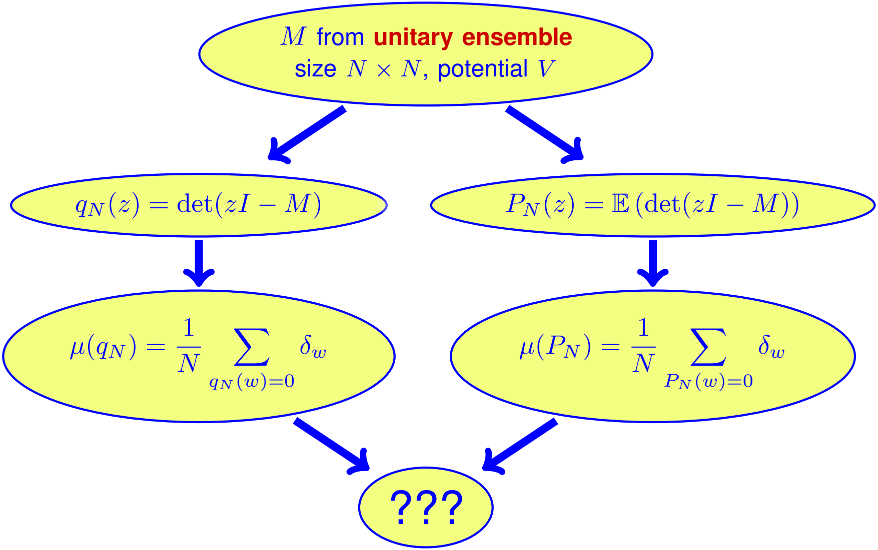
Furthermore, for some $h_N>0$,

$$\frac{1}{h_N}p_N(x)=P_N(x)=\mathbb{E}\left[\det(Ix-M)\right]$$

Main message: *all information is encoded in the OP's!*

Back to our original question



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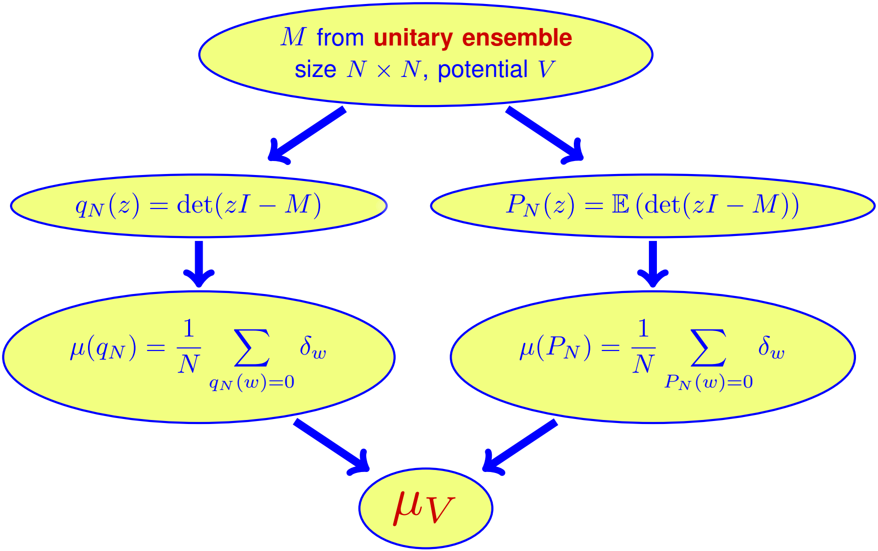

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M from **unitary ensemble**
size $N \times N$, potential V

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$$\mu(P_N) = \frac{1}{N} \sum_{P_N(w)=0} \delta_w$$

$$\mu_V$$

OP's are the unifying principle!

The normal matrix model

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- Normal matrix model = space of $N \times N$ normal random matrices ($MM^* = M^*M$) with probability distribution of the form

$$\propto \exp\left(-\frac{N}{t_0}\,\mathrm{Tr}\,\mathcal{V}(M)\right)dM$$

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- This distribution can again be expressed in terms of OP's, but now for the planar measure $e^{-\frac{N}{t_0}\mathcal{V}(z)}dA(z)$

The cut-off approach for algebraic potentials

- For the potential

$$\mathcal{V}(z) = |z|^2 - 2\,\mathrm{Re}\,V(z), \quad V(z) = \sum_{k=1}^{d} \frac{t_k}{k} z^k,$$

  the NMM is connected to Laplacian growth and quadrature domains (Kostov, Krichever, Mineev-Weinstein, Wiegmann and Zabrodin, 2001)

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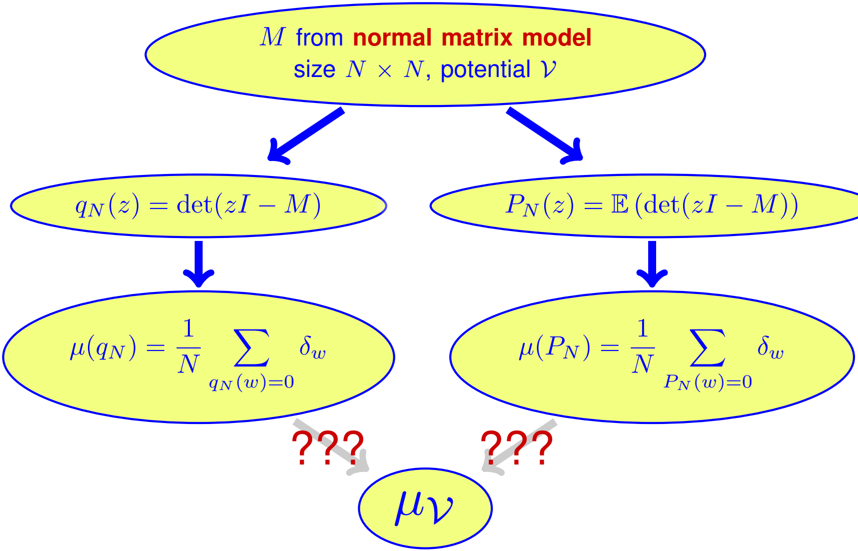
- Instead of considering all normal matrices, Elbau & Felder proposed to consider normal matrices with eigenvalues restricted to lie within a compact $D \subset \mathbb{C}$
- At the end of the day, eigenvalue statistics are expected to be independent of specific geometry of D (at least for small t_0)

Eigenvalue distribution [Elbau & Felder, 2005]



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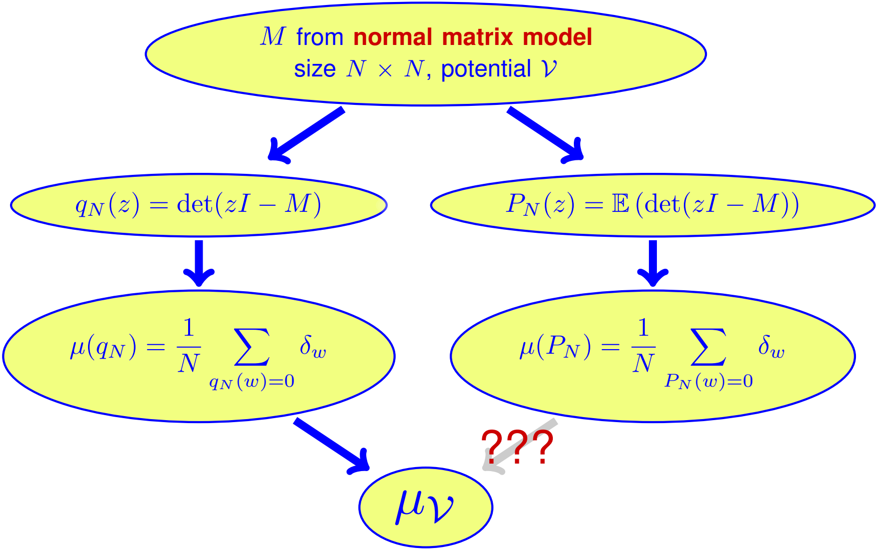

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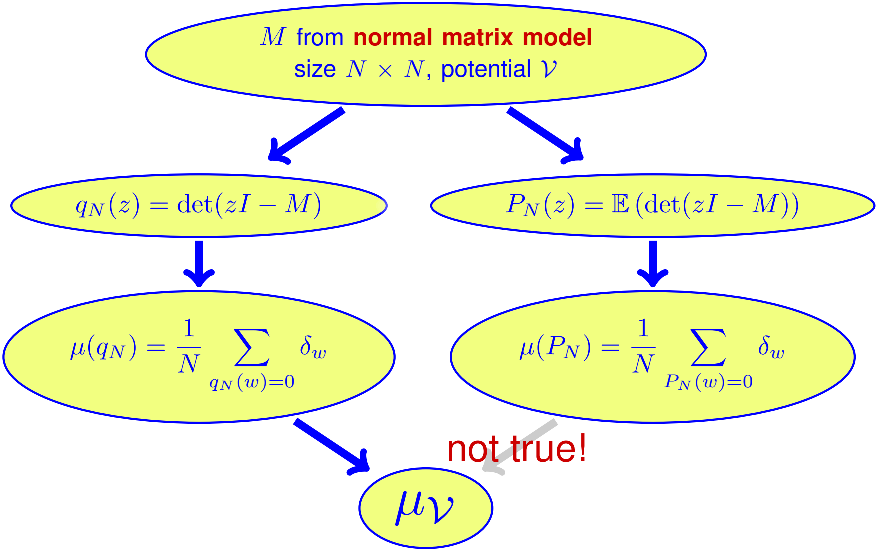

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  - $\displaystyle \int \log \frac{1}{|s-z|} d\mu_{\mathcal{V}}(s) \leq \int \log \frac{1}{|s-z|} d\mu_*(s), \quad z \in \Omega$
- Given $\mu_{\mathcal{V}}$, the existence of μ_* is highly nontrivial!

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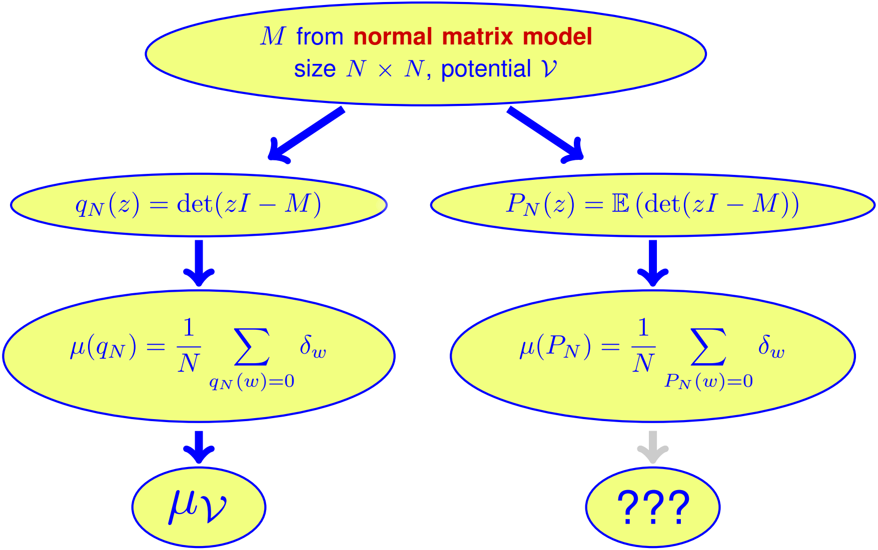

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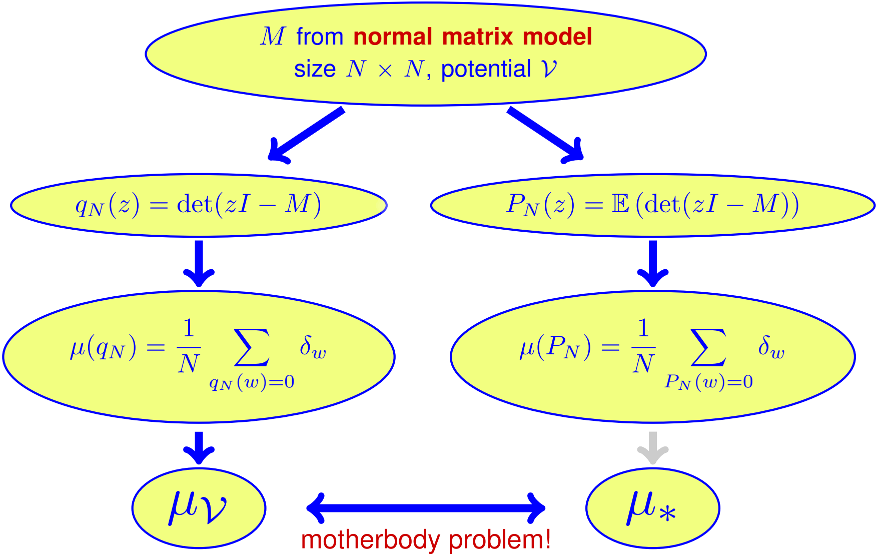

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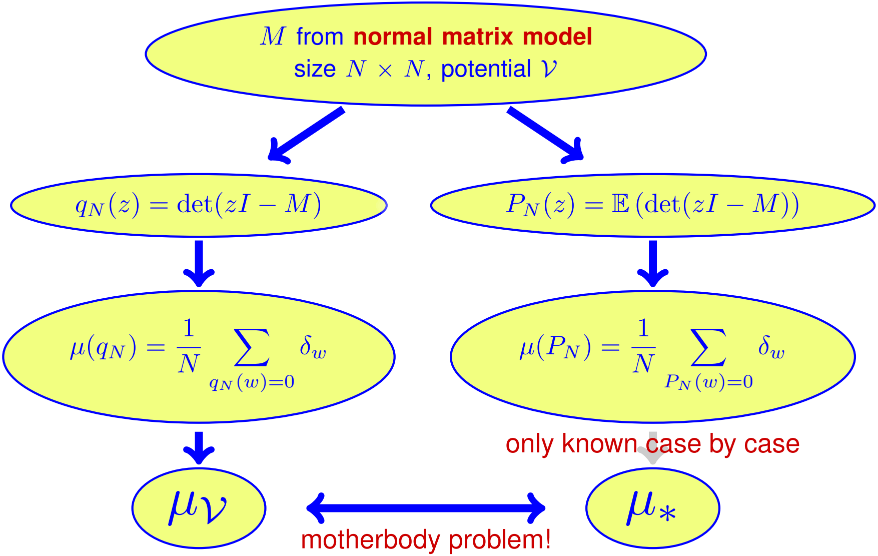

The cubic potential

For now on, we specify to

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- Symmetric case $t_1 = 0$ studied by Bleher & Kuijlaars (2012)

Mean eigenvalue distribution - computation

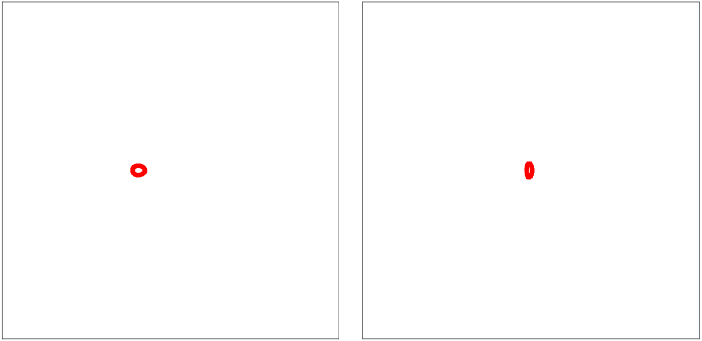
Theorem (Bleher & S., 2017, to appear)

There exists $t_{0,crit} = t_{0,crit}(t_1) > 0$ *for which*

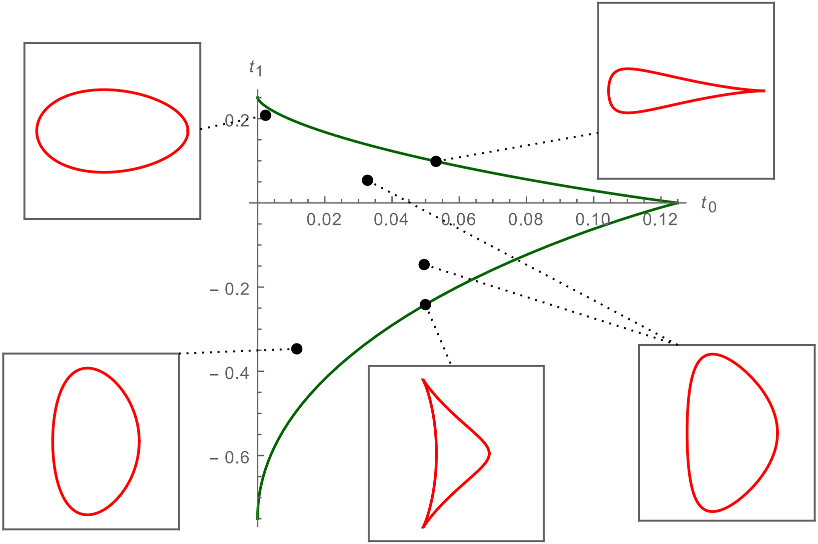
$$d\mu_{\mathcal{V}}(z) = \frac{1}{\pi t_0} \chi_\Omega(z) dA(z), \quad 0 < t_0 < t_{0,crit}$$

and Ω *can be explicitly computed (through algebraic conditions on* t_0 *and* t_1*)*

Evolution of the boundary for $t_1 = \frac{1}{16}$ (left) and $t_1 = -\frac{1}{4}$ (right)



Phase diagram



The mother body phase transition

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The mother body phase transition - $t_1 = 1/5$

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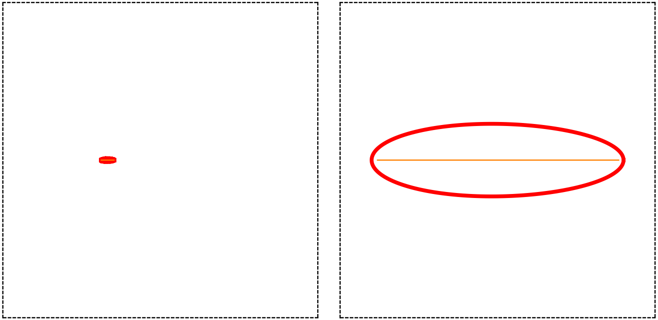
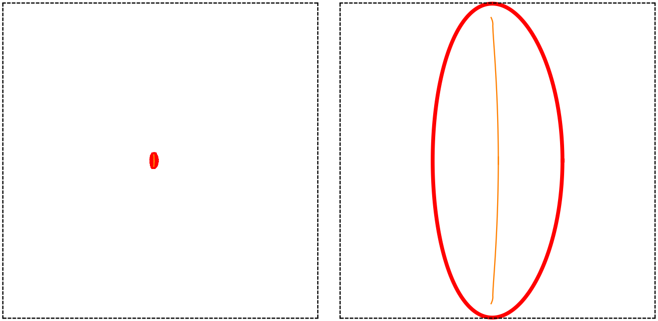


The mother body phase transition - $t_1 = -1/4$

Theorem (Bleher & S., 2017, to appear)

For $t_1 \in (-3/4, 1/4)$, the measure $\mu_{\mathcal{V}}$ admits a mother body μ_*.



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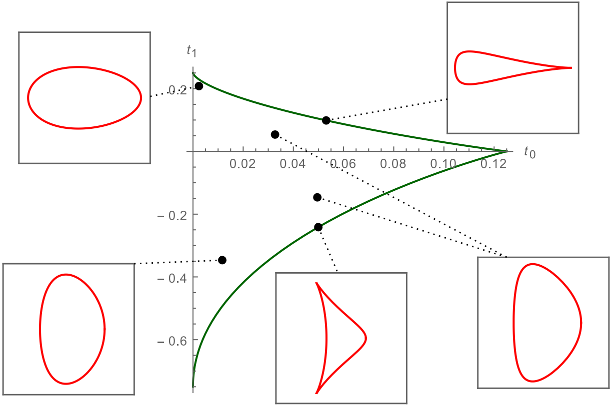

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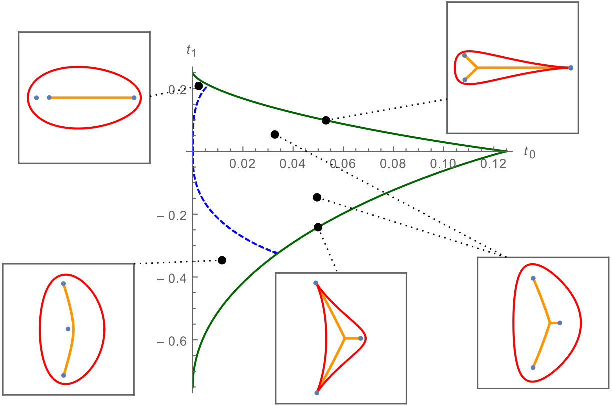

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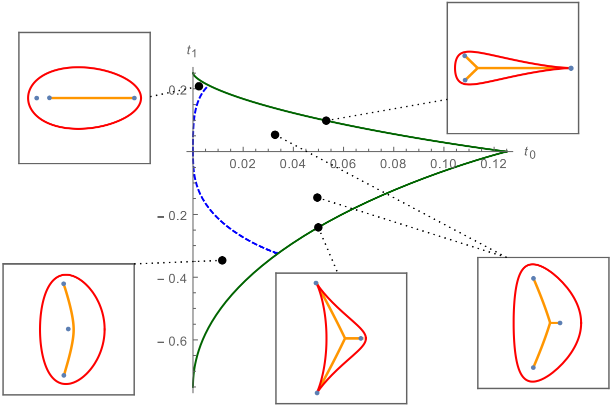

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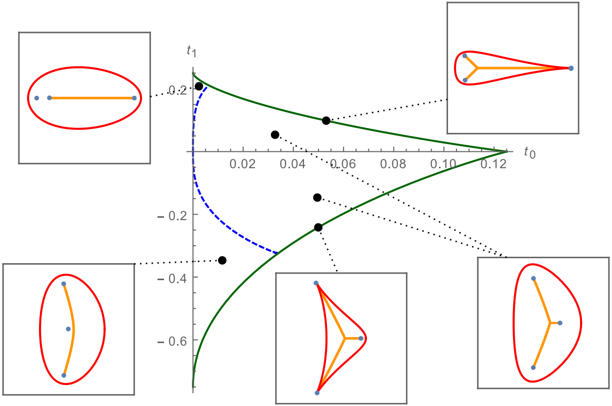

- We also verify the convergence $\mu(P_N) \xrightarrow{*} \mu_*$ for a regularized P_N
- So in words, the eigenvalues are not sensitive to the phase transition of the zeros of P_N

Some words on the proof

- For some $A = A(t_0, t_1)$ and $B = B(t_0, t_1)$, the pairs of points of the form $(\xi, z) = (h(w^{-1}), h(w)),\ w \in \mathbb{C}$ satisfy an algebraic equation (a.k.a. spectral curve) of the form

$$\xi^3 + z^3 - \xi^2 z^2 - t_1(\xi^2 + z^2) - (1 + t_0)\xi z + B(\xi + z) + A = 0$$

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- Construct a quadratic differential ϖ on the associated Riemann surface $\mathcal{R}$ and describe its critical graph $\mathcal{G}$ for $t_1 = 0$
- For $t_1 = 0$, embed μ_* on $\mathcal{G}$
- Deform $\mathcal{G}$ with parameter t_1, keeping track of μ_*

Thank you!