Intrinsic supersmoothness of piecewise multivariate functions

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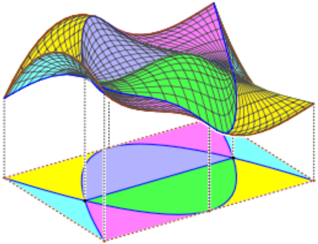
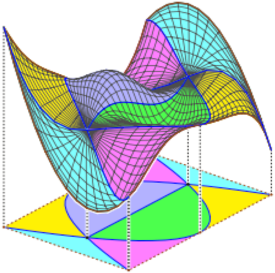
C^r (smooth) piecewise polynomials of degree $\leq d$ over (simplicial) partitions Δ in $\mathbb{R}^n$ are called *SPLINES*. It is a vector space $S^r_d(\Delta)$.





From "BIVARIATE SEMIALGEBRAIC SPLINES" by MICHAEL DIPASQUALE AND FRANK SOTTILE

splines on triangulations

A set $\Delta = \{T_1, \ldots, T_N\}$ of triangles in the plane is called a **triangulation** of $\Omega = \bigcup_{i=1}^{N} T_i$ provided that

1. If a pair of triangles in Δ intersect, then that intersection is either a common vertex or a common edge.
2. The domain Ω is homeomorphic to a disk.

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2. The domain Ω is homeomorphic to a disk.

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s\big|_{T_i} \in P_d, \quad i = 1, \ldots, N\}$$

where P_d is now the $\binom{d+2}{2}$-dimensional space of polynomials of degree d in two variables.

why do we use splines

approximation theory: given some information $I(f)$ about a function f from a certain class, build a spline $s(f)$ which is sufficiently close to f in a certain norm.

numerical PDEs: given a PDE and some boundary conditions, build a spline s which is sufficiently close to the solution of the PDE in a certain norm.

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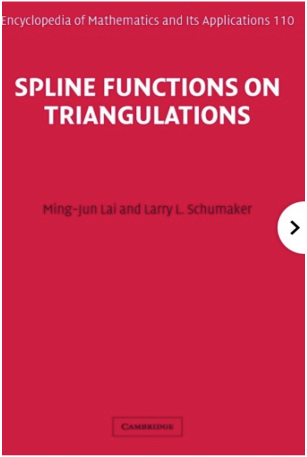
- interpolation: $I(f) = I(s)$, values, derivatives at faces of Δ
- approximation: $\|f - s(f)\| \le K|\Delta|^m\|f\|$, Δ is the "mesh size" of Δ

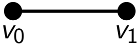
everything about splines except what is in this talk



… collaborators

Boris Shekhtman
University of South Florida

univariate splines: dimension count

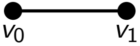


$$\Delta_0 = [v_0, v_1]$$

$$a_0 + a_1 x + a_2 x^2$$

$$\dim\ S_2^1(\Delta_0) = 3$$

univariate splines: dimension count



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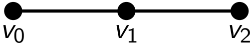
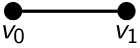


$$\Delta_1 = [v_0, v_1] \cup [v_1, v_2]$$
$$\dim S_2^1(\Delta_1) =$$
$$= 3 + 3 - 2 = 4$$

univariate splines: dimension count



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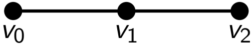

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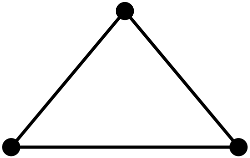
$$\dim\ S_2^1(\Delta_1) =$$

$$= 3 + 3 - 2 = 4$$

$$\dim S_2^1(\Delta_n) = \#\ \text{parameters - }\#\ \text{conditions} =$$

$$= 3(n+1) - 2n = n + 3$$

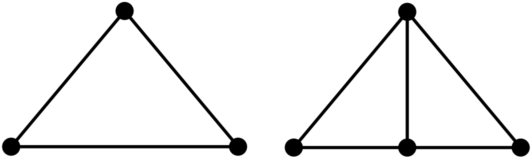
bivariate splines: dimension count



$\sum_{i+j\leq 2} a_{ij}x^i y^j$

dim $S_2^1(\Delta_0) = 6$

bivariate splines: dimension count



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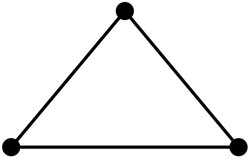
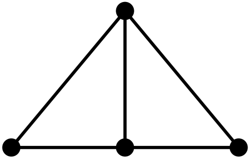
dim $S_2^1(\Delta_0) = 6$

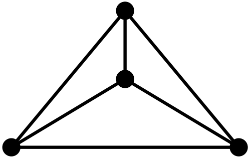
dim $S_2^1(\Delta_1) =$

$6 + 6 - (3 + 2) = 7$

bivariate splines: dimension count







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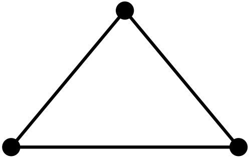
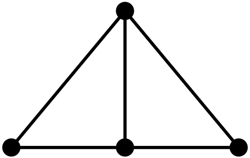
bivariate splines: dimension count



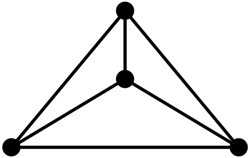
$\sum_{i+j\le 2} a_{ij}x^i y^j$

dim $S_2^1(\Delta_0) = 6$



dim $S_2^1(\Delta_1) =$

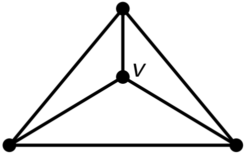
$6 + 6 - (3 + 2) = 7$



dim $S_2^1(\Delta_2) = 6$

$\sum_{i+j\le 2} a_{ij}x^i y^j$

first example of supersmoothness: Clough-Tocher split



$$\boxed{S_d^1(\Delta_{CT}) = S_d^1(\Delta_{CT}) \cap C^2(v)}$$

$S_d^1(\Delta_{CT})$ has instrinsic supes-moothness two at v

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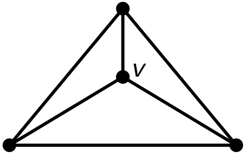

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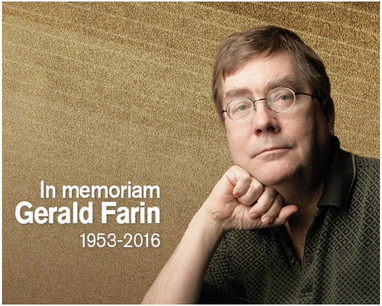

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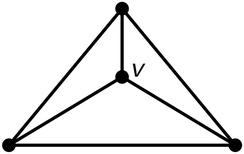

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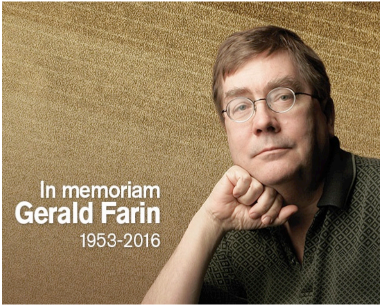
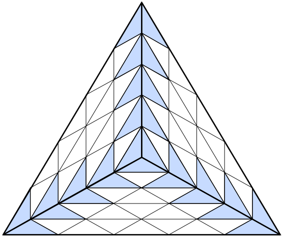


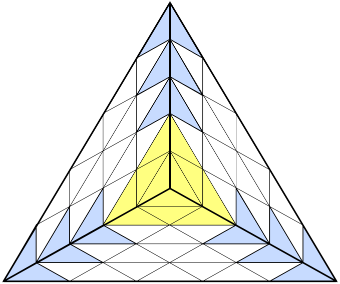
a non-mathematical analogy

If one picture is worth a thousand words, then two should suffice to explain the statement of a research problem. Imagine three triangular pieces of your favorite material being joined together along three edges. How smoothly do the pieces fit together? Let us use the scale from zero to five, where zero is not very smooth, and five means the fit is so smooth that we cannot see any edges at all. The blue "stitching" has shaded width one, i.e., the joint on the left has smoothness one according to our scale.



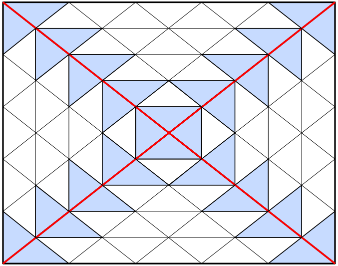
a non-mathematical analogy: supersmoothness

We now state the unexplainable: the joint has higher smoothness at the center where all three edges meet. If we use our scale, this "supersmoothness" is two at the center, see the wider yellow shaded "stitching" at the center.



a non-mathematical analogy: supersmoothness

It becomes more intriguing: if we use four pieces instead of three (imagine a square instead of a triangle), there is no additional smoothness at the center.



supersmoothness of splines on cells

- Splits of triangles, squares and other polygons using one interior point are called bivariate cells. Splits of prisms, cubes and other solids using one interior point are called trivariate cells. More difficult to imagine and more important in applications are cells in higher dimensions.
- Suppose Δ is a simplicial partition consisting of a set of simplices which all share one common interior vertex. Then we call Δ a cell.
- Is supersmoothness at the center of the cell an algebraic (polynomial) or an analytic (smooth functions) property?
- Farin's and Alfeld's proofs (1972 and 1983) were essentially analitic.
- However, the correct answer is ... both !

supersmoothness of bivariate splines on cells

Theorem (T.S. 2010)

Let Δ be a cell with n noncollinear edges. Then

$$S_d^r(\Delta) = S_d^r(\Delta) \cap C^\rho(v), \quad \textit{where} \quad \rho = r + \left\lfloor \frac{r+1}{n-1} \right\rfloor.$$

Example 1. $n = 3$, $r = 1$, $d = 6$, $\rho = 2$ and $S_6^1(\Delta) = S_6^1(\Delta) \cap C^2(v)$

Example 2. $n = 3$, $r = 3$, $d = 6$, $\rho = 5$ and $S_6^3(\Delta) = S_6^3(\Delta) \cap C^5(v)$

supersmoothness of bivariate splines on cells

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Example 2. $n = 3$, $r = 3$, $d = 6$, $\rho = 5$ and $S_6^3(\Delta) = S_6^3(\Delta) \cap C^5(v)$

There is a big difference between Ex. 1 and Ex. 2: compare r and ρ!

supersmoothness of bivariate splines on cells: true C^r ?

Theorem (T.S. 2010)

Let Δ be a cell with n noncollinear edges. Then

$$S_d^r(\Delta) = S_d^r(\Delta) \cap C^\rho(v), \quad \textit{where} \quad \rho = r + \left\lfloor \frac{r+1}{n-1} \right\rfloor,$$

where $C^\rho(v)$ is understood as matching of the derivatives up to order ρ at the point v if $r+2 \leq \rho < d$.

Ex 1. $n = 3$, $r = 1$, $d = 6$, $\rho = 2$ and $S_6^1(\Delta) = S_6^1(\Delta) \cap C^2(v)$. True C^2 at v. Analytic proof is possible.

Ex. 2. $n = 3$, $r = 3$, $d = 6$, $\rho = 5$ and $S_6^3(\Delta) = S_6^3(\Delta) \cap C^5(v)$. Matching of the derivatives at v. Only algebraic proof is possible.

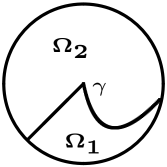
supersmoothness at singular point

Theorem (T.S., B. Shekhtman 2012)

Let $\gamma \subset \mathbb{R}^2$ be the trace of a Jordan arc that divides the open disk Ω into two subsets Ω_1 and Ω_2. Let γ be not smooth at $P \in \gamma$. Let f_1, f_2 be C^1 functions on Ω continuously glued along γ, that is, let

$$F(x,y) := \begin{cases} f_1(x,y) & \text{if} \quad (x,y) \in \Omega_1, \\ f_2(x,y) & \text{if} \quad (x,y) \in \Omega_2, \end{cases}$$

be a continuous function on Ω. Then the piecewise function F is differentiable at P, that is, $\nabla f_1(P) = \nabla f_2(P)$.



local characterization of non-smooth curves

Theorem (T.S., B. Shekhtman 2012)

The trace of a Jordan arc γ is smooth at P if and only if there exists a neighborhood U of P and a function h continuously differentiable on U such that

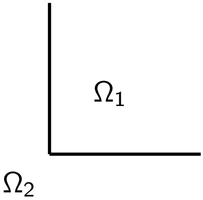
$$h(x,y) = 0 \quad \text{if} \quad (x,y) \in \gamma \cap U, \quad \text{and} \quad \nabla h(P) \neq \mathbf{0}.$$

proof of a special case

Let $p_1(x, y)$ and $p_2(x, y)$ be polynomials, and let

$$s(x,y) := \begin{cases} p_1(x,y) & \text{if} \quad (x,y) \in \Omega_1 \\ p_2(x,y) & \text{if} \quad (x,y) \in \Omega_2 \end{cases}$$

be a continuous function on Ω.



$$p_1(x,y) = a_{00} + a^1_{10}x + a^1_{01}y + \cdots$$

$$p_2(x,y) = a_{00} + a^2_{10}x + a^2_{01}y + \cdots$$

$$p_1(0,y) = p_2(0,y) \quad \Rightarrow \quad a^1_{01} = a^2_{01}$$

$$p_1(x,0) = p_2(x,0) \quad \Rightarrow \quad a^1_{10} = a^2_{10}$$

$$\nabla p_1(0,0) = \nabla p_2(0,0).$$

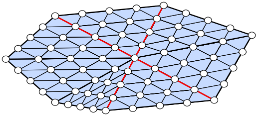
supersmoothness across edges

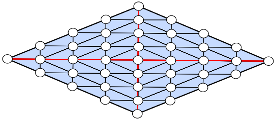
Theorem

Let Δ be a cell, and let smoothness $r \geq 1$. Suppose the number of different slopes $m \leq r+2$. Let $\widetilde{\Delta}$ be the cell obtained from Δ by removing the edges with no collinear counterparts. Then

$$S^r_{r+1}(\Delta) = S^r_{r+1}(\widetilde{\Delta}).$$

Example: $r = 3$. Three black edge can be removed. Alfeld's applet.





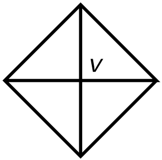
mixed derivatives

Theorem (T.S. 2012)

Let Δ be a cell with no non-collinear and 2ℓ collinear edges meeting at v. Then for any $s \in S_d^{\ell-1}(\Delta)$ any ℓ-th order mixed derivative

$$\frac{\partial^\ell s}{\partial u_{i_1} \cdots \partial u_{i_\ell}}(v),$$

where $u_{i_1}, \ldots, u_{i_\ell}$ are pairwise distinct directions of non-collinear edges, exists.

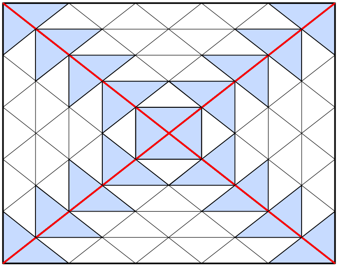


$$s \in S_d^1(\Delta), \quad \ell = 2$$

$$\frac{\partial^2 s}{\partial x \partial y}(v) \quad \text{exists}$$

a non-mathematical analogy: supersmoothness

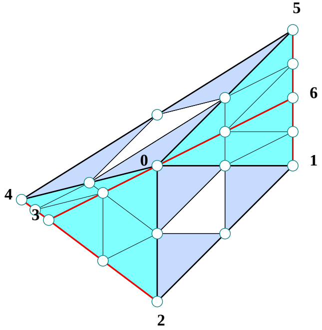
It becomes more intriguing: if we use four pieces instead of three (imagine a square instead of a triangle), there is no additional smoothness at the center, but the mixed derivatives match!

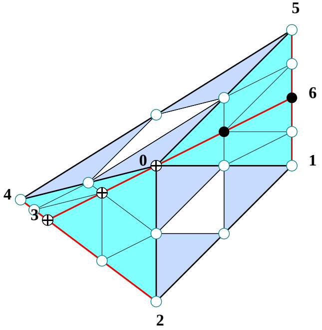


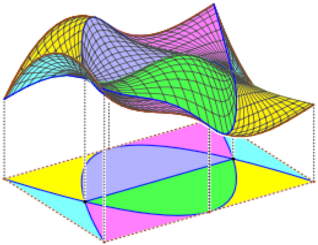
an exotic type of supersmoothness

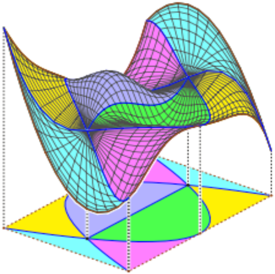
Theorem (T.S. 2012)

Let $\triangle$ be a cell with four non-collinear edges meeting at the point v. Then there exists a unique straight line passing through v with the property that for any smooth quadratic spline s on $\triangle$, the restriction of s on this line is a univariate quadratic polynomial.









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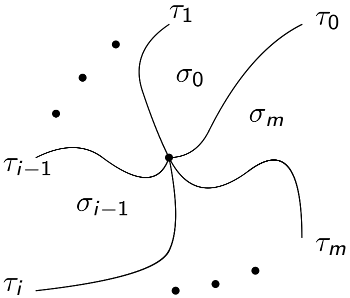
supersmoothness of semialgebraic splines

Let Δ be a cell complex with one interior vertex defined by irreducible algebraic arcs $\{\tau_i\}_{i=0}^m$



Theorem (T.S, B. Shekhtman, 2022)

Let $\{\tau_{i_j}\}_{j=0}^{n}$, $1 \leq n \leq m$, be the edges of Δ defined by the polynomials $\{g_{i_j}\}_{j=0}^{n}$ with non-zero gradients at the origin. If $\{\nabla g_{i_j}(0,0)\}_{j=0}^{n}$ are pairwise linearly independent, then every $s \in S_d^{n-1}(\Delta)$ has supersmoothness n at the origin. If $n = 0$, or if all edges in Δ are defined by polynomials with zero gradients, then every $s \in S_d^0(\Delta)$ has supersmoothness 1 at the origin.

why study supersmoothness

- Choice of suitable triangulations.
- Dimension of bivariate splines on arbitrary triangulations is a hard open problem. Knowing intrinsic supersmoothness provides sharper lower bounds.
- Intrinsic supersmoothness directly affects interpolating properties: restricts the choices of interpolating sets.
- Numerical PDEs: prediction of (usually undesirable) extra smoothness of corner finite elements.
- Intrinsic supersmoothness might affects local convergence order.

historical developments

- G. Farin, Bézier polynomials over triangles; Report TR/91, Dept. of Mathematics, Brunel University, Uxbridge, UK, 1980
- P. Alfeld, A trivariate Clough-Tocher scheme for tetrahedral data, *Computer Aided Geometric Design* **1**, 1984 169–181
- T. Sorokina, Intrinsic supersmoothness of multivariate splines, *Numerische Mathematik*, **116**, 2010, 421–434

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After 2010:

Analytic: B. Shekhtman, M. Floater, K. Hu, T. Sorokina

Algebraic: M. DiPasquale, N.Villamizar, B. Yuan, T. Sorokina,
D. Toshniwal, H. Schenck

key references and thank you for listening!

- Floater, M.S., Hu, K. A characterization of supersmoothness of multivariate splines, *Adv Comput Math*, 2020.
- Shekhtman B., and T. Sorokina, A note on intrinsic supersmoothness of bivariate semialgebraic splines, *Computer Aided Geometric Design*, 2022.
- Shekhtman B., and T. Sorokina,A, Intrinsic Supersmoothness, *Journal of Concrete and Applicable Mathematics*, 2015
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the most famous conjecture

$S_3^1(\Delta)$ smooth cubic splines in two variables on a triangulation Δ

V_B number of boundary vertices in Δ

V_I number of interior vertices in Δ

σ_{sing} number of vertices where four edges meet with two slopes

$$\dim\ S_3^1(\Delta) = 3V_B + 2V_I + 1 + \sigma_{sing}$$

known results on dim

If Δ_2 is shellable, then dim $S^r_d(\Delta_2)$ is bounded above by

$$\binom{d+2}{2} + E_I\binom{d+1-r}{2} - V_I\left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \sum_{v\in\mathcal{V}_I} \sigma_v,$$

where E_I is the number of interior edges, V_I is the number of interior vertices, $\mathcal{V}_I$ is the set of interior vertices of Δ_2, and

$$\sigma_v := \sum_{j=1}^{d-r} (r+j+1-jm_v)_+, \quad m_v := \text{ number of different slopes at } v$$

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If $d \geq 3r+1$, the upper bound $=$ dim

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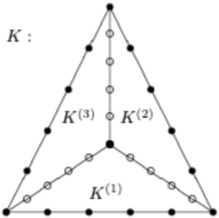
If $d \geq 3r+1$, the upper bound = dim

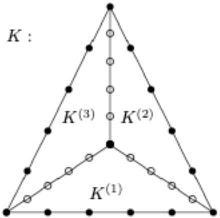
Not much is known for $d \leq 3r$

collaborators on Linear Differential Operators on Splines

Peter Alfeld

Shangyou Zhang


K :
$K^{(3)}$
$K^{(2)}$
$K^{(1)}$



Theorem 2.1. *Let* $\{z_m\}_{m=1}^{3k+1}$ *be arbitrary real numbers, let* K *be a triangle with three angles* $\theta_1 \geq \theta_2 \geq \theta_3$. *There exists a unique* $u_h \in V_h$ *satisfying the interpolation conditions*

$$u_h(\mathbf{y}_i) = z_m, \quad \forall \quad \mathbf{y}_m \in \mathcal{I}_K, \tag{4}$$

if the three angles of K *satisfy the constraints*

$$\begin{aligned} t_j(\theta_1, \theta_2) := & \frac{p_{2j+1}(-2\cot\theta_1 - \cot\theta_2, 1)}{p_{2j+1}(\cot\theta_1 + 2\cot\theta_2, 1)} \cdot \frac{p_{2j+1}(-2\cot\theta_2 - \cot\theta_3, 1)}{p_{2j+1}(\cot\theta_2 + 2\cot\theta_3, 1)} \\ & \cdot \frac{p_{2j+1}(-2\cot\theta_3 - \cot\theta_1, 1)}{p_{2j+1}(\cot\theta_3 + 2\cot\theta_1, 1)} \neq 1, \quad j = 2, ..., k, \end{aligned} \tag{5}$$

where

$$p_{2j+1}(x, y) = \operatorname{Im}\left((x + iy)^j\right).$$