On Lagrange polynomials in approximating planar sets by Julia sets

Leokadia Bialas-Ciez (Poland), Marta Kosek (Poland), Margaret Stawiska-Friedland (USA)

JAGIELLONIAN UNIVERSITY IN KRAKOW

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Interpolation polynomials

Let $n \in \mathbb{N}$ and $E \subset \mathbb{C}$ be compact. Consider an array A of distinct points in E, i.e. $A = \{\zeta^{(n)} : n \geq 1\}$ where $\zeta^{(n)} = \{\zeta_0^{(n)}, ..., \zeta_n^{(n)}\}$, $n \in \mathbb{N}$, is an $(n+1)$-tuple of distinct points in E. Fix $j \in \{0, 1, ..., n\}$. We put

- $V(\zeta^{(n)}) = \prod_{0 \leq l < k \leq n} \left|\zeta_l^{(n)} - \zeta_k^{(n)}\right|$.
- $\Delta^{(j)}(\zeta^{(n)}) := \prod_{k \in \{0,...,n\} \setminus \{j\}} \left(\zeta_j^{(n)} - \zeta_k^{(n)}\right)$;
- $L^{(j)}(z, \zeta^{(n)}) := \left(\Delta^{(j)}(\zeta^{(n)})\right)^{-1} \cdot \prod_{k \in \{0,...,n\} \setminus \{j\}} \left(z - \zeta_k^{(n)}\right)$.

$L^{(j)}(\cdot, \zeta^{(n)})$ is the j-th fundamental Lagrange interpolation polynomial with nodes $\zeta^{(n)}$.

Some convergence results

Let $A = \{\zeta^{(n)} : n \geq 1\}$ be an array of distinct points in a compact set E. The n-th Lebesgue constant is defined by the formula

$$\Lambda_n(E, \zeta^{(n)}) := \max_{z \in E} \sum_{j=0}^{n} |L^{(j)}(z, \zeta^{(n)})|.$$

Let E be polynomially convex regular compact set and

$$\lim_{n \to \infty} \Lambda_n(E, \zeta^{(n)})^{1/n} = 1. \qquad (1)$$

This condition implies nice interpolation properties, e.g. if f is holomorphic in a neighbourhood of E, its interpolation polynomials with nodes $\zeta^{(n)}$ converge uniformly (and fast) on E to f.

Main Theorem 1 ([BCKS]) *Let E be a polynomially convex regular compact set in $\mathbb{C}$ and $A = \{\zeta^{(n)} : n \geq 1\}$ an array of distinct points in E. Then (??) is equivalent to*

$$\lim_{n \to \infty} \min_{k \in \{0,...,n\}} |\Delta^{(k)}(\zeta^{(n)})|^{1/n} = \operatorname{cap} E. \qquad (2)$$

Fix $j = j_n \in \{0, ..., n\}$ such that $|\Delta^{(j_n)}(\zeta^{(n)})| = \min_k |\Delta^{(k)}(\zeta^{(n)})|$. Put $L_n := L^{(j_n)}(\cdot, \zeta^{(n)})$.

Theorem 2 ([BCKS]) *Let E be a polynomially convex regular compact subset of $\mathbb{C}$ and ω be a modulus of continuity of g_E. Let $\{\zeta^{(n)} : n \geq 1\}$ be an array of distinct points from E such that (??) holds and let L_n be defined as above. If $\operatorname{dist}(z, E) \geq 1/n^2$ and $\Theta(E) := (2 + \operatorname{diam} E)^{1/3}$, then*

$$\frac{1}{n} \log \frac{1}{||L_n||_E} \leq g_E(z) - \log \sqrt[n]{|L_n(z)|} \leq \frac{3}{n} \log[(n+1)\Theta(E)] + \omega\left(\frac{1}{n^2}\right).$$

Moreover, the expression on the left hand side tends to 0.

Pseudo Leja sequences

Definition (Bialas-Ciez, Calvi (2012)) Let $(C_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n \to \infty} (C_n)^{1/n} = 1$. A sequence $(a_n)_{n=0}^{\infty}$ of points in E is said to be a pseudo Leja sequence (of Edrei growth (C_n)) if a_0 is an arbitrary point of E and a_n is such a point in E that

$$C_n\, |w_n(a_n)| \geq \max_{z \in E} |w_n(z)|, \qquad w_n(z) := \prod_{j=0}^{n-1} (z - a_j).$$

For $C_n \equiv 1$ we get a classical Leja sequence.

Theorem 3 ([BCKS]) *Let E be compact such that $\partial \widehat{E}$ is a finite union of $\mathcal{C}^2$ arcs. Let $(a_n)_{n=0}^{\infty}$ in $\partial \widehat{E}$ be a pseudo Leja sequence of bounded Edrei growth (C_n). Then $\lim_{n \to \infty} \Lambda_n(E, a^{(n)})^{1/n} = 1$, where $a^{(n)} = \{a_0, ..., a_n\}$.*

The reference and contact details

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UJ=Jagiellonian University, Kraków; MR=Mathematical Reviews

e-mails of the corresponding authors:
. Marta.Kosek@im.uj.edu.pl, stawiska@umich.edu

The Green function

Let $E \subset \mathbb{C}$ be compact. The polynomially convex hull of E is

$$\widehat{E} = \left\{ z \in \mathbb{C} : |p(z)| \leq \|p\|_E := \max_{w \in E} |p(w)| \text{ for every polynomial } p \right\}.$$

E is polynomially convex if $E = \widehat{E}$.

E is regular if its Green function g_E exists. This is the unique function such that $g_E \equiv 0$ on $\widehat{E}$; g_E is harmonic and positive in $\mathbb{C} \setminus \widehat{E}$; $g_E(z) \to 0$ as $z \to \partial D$ and $g_E(z) - \log|z|$ tends to a finite number γ as $z \to \infty$. It is known that $\operatorname{cap} E = \exp(-\gamma)$ and

$$g_E = \log \Phi_E(z) \quad \text{with} \quad \Phi_E(z) := \sup_p |p(z)|^{1/\deg p}, \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all non-constant polynomials such that $||p||_E \leq 1$.

Approximation of planar sets by Julia sets

If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial then

$$\mathcal{K}(P) := \{z \in \mathbb{C} : (P^n(z))_{n=1}^{\infty} \text{ is bounded}\}$$

is its filled-in Julia set and $\mathcal{J}(P) = \partial \mathcal{K}(P)$ its Julia set. If $\deg P \geq 2$, then $\mathcal{K}(P)$ is a polynomially convex regular compact set.

Theorem 4 (Lindsey, Younsi (2016)) *Let $E \subset \mathbb{C}$ be any nonempty compact set with connected complement. Then for any $\varepsilon > 0$ there exists a polynomial P such that*

$$\chi(E, \mathcal{K}(P)) < \varepsilon, \quad \chi(\partial E, \mathcal{J}(P)) < \varepsilon,$$

where χ denotes the Hausdorff metric.

On the approximation rate in the Klimek metric

Let $E, F \subset \mathbb{C}$ be regular. We define Klimek's metric $\Gamma(E, F) := \max(||g_E||_F, ||g_F||_E)$ (the family of all polynomially convex regular sets with Γ is a complete metric space).
Assume that $0 \in E$. Let $s_n = (3/n) \log((n+1)/(\operatorname{diam} E + 2))$ and let $\eta^{(n)}$ be a Fekete (extremal) $(n+1)$-tuple for $\{z \in \mathbb{C} : g_E(z) \leq s_n\}$ (i.e., an $(n+1)$-tuple maximizing $V(\zeta^{(n)})$ in this set). We order the points of $\eta^{(n)}$ so that

$$|\Delta^{(0)}(\eta^{(n)})| \leq \min_{j \in \{1,...,n\}} |\Delta^{(j)}(\eta^{(n)})|.$$

Proposition 5 ([BCKS]) *Let E be a compact subset of $\mathbb{C}$ such that g_E is Hölder continuous with the constant A and the exponent $\alpha \in (0, 1]$. Put*

- $P_n(z) = z(n+1)^{-(A+4)} L^{(0)}(z, \eta^{(n)})$ *if* $\alpha \in [1/2, 1]$;
- $P_n(z) = z \exp\left(-(A+4)n^{1-2\alpha}\right) L^{(0)}(z, \eta^{(n)})$ *if* $\alpha \in (0, 1/2)$.

Then for n large enough,

- $\Gamma(E, \mathcal{K}(P_n)) \leq (3A + 12)n^{-1} \log(n+1)$ *if* $\alpha \in [1/2; 1]$;
- $\Gamma(E, \mathcal{K}(P_n)) \leq (3A + 12)n^{-2\alpha}$ *if* $\alpha \in (0; 1/2)$.

On the approximation rate in the Hausdorff metric

We say that a regular compact set E satisfies the Łojasiewicz-Siciak condition if for any bounded neighbourhood U of E there exist positive constants B, β such that

$$g_E(z) \geq B(\operatorname{dist}(z, E))^{\beta}, \qquad z \in U.$$

Proposition 6 ([BCKS]) *Let E be a compact set satisfying the Łojasiewicz-Siciak condition. Assume also that the Green function g_E is Hölder continuous with exponent $\alpha \in (0, 1]$.
Then there exist positive constants C, κ, depending only on the set E and, for each $n \in \mathbb{N}$, large enough, a polynomial P_n of degree $n+1$ such that*

- $\chi(E, \mathcal{K}(P_n)) \leq C\left(n^{-1} \log(n+1)\right)^{\kappa}$ *if* $\alpha \in [1/2; 1]$;
- $\chi(E, \mathcal{K}(P_n)) \leq C n^{-2\alpha\kappa}$ *if* $\alpha \in (0; 1/2)$.

One can take here polynomial P_n from Proposition **??**. Instead of Fekete points in E it is also possible to use a suitable pseudo Leja sequence in the level set $\{z \in \mathbb{C} : g_E(z) = \tau_n\}$ (if τ_n is small and appropriately chosen).