Signal decomposition via SuperEMD

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Introduction

- Stationary signal:

$$f(t) = a_0 + \sum_{j=1}^{N} a_j \cos(2\pi(\omega_j t + \eta_j)), \quad t \in \mathbb{R},$$

for arbitrary real values $a_j > 0$, η_j and $\omega_j > 0$ (*frequency*).

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- Non-stationary signal:
Adaptive harmonic model (AHM):

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where $A_j(t) > 0$ and each $\phi_j(t)$ is a general C^1 function such that $\phi_j'(t) > 0$ (*instantaneous frequency*).

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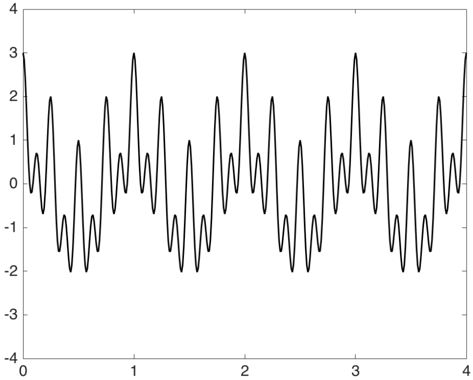
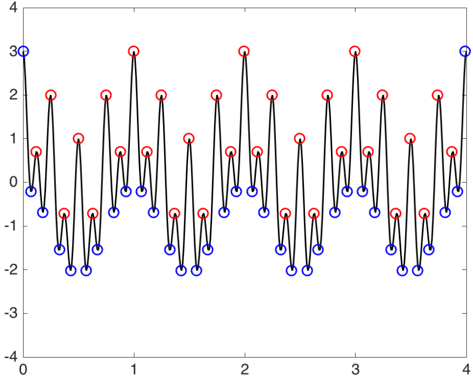
$$f(t) = A_0(t) + \sum_{j=1}^{N} A_j(t)\cos(2\pi\phi_j(t))$$

- EMD: Empirical mode decomposition
- Introduced by Norden Huang and others (1998)
- Data-driven way to analyze non-stationary signals

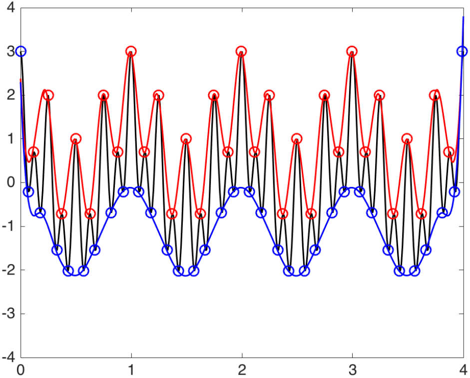
- Given a signal $f(t)$. Set $h_{1,0} := f$.



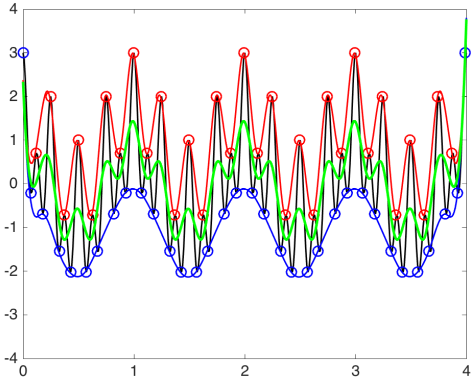
- Find its local maxima and minima.



- Compute the upper and lower envelopes through (standard) cubic spline interpolation of the extrema.



- Compute the mean envelope $m_{1,1}$.



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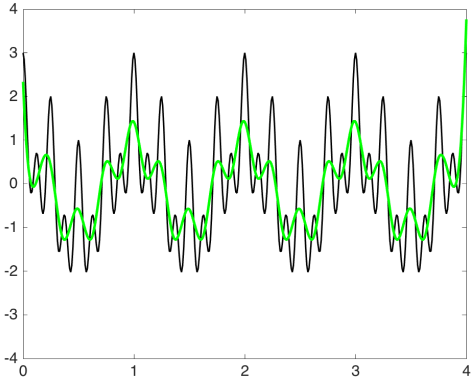
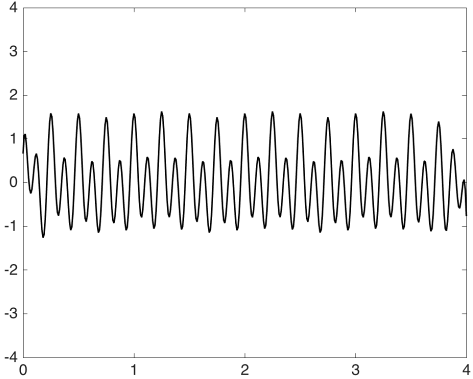
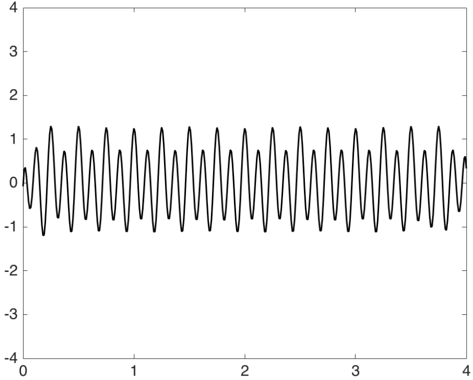


- Subtract the mean envelope from $h_{1,0}$.



$h_{1,1} := h_{1,0} - m_{1,1}$

- Repeat this process until $h_{1,\ell}$ is an *intrinsic mode function* (IMF).



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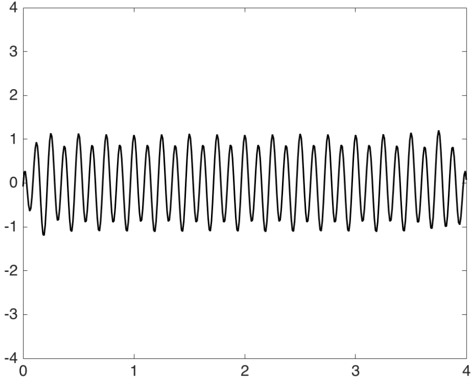

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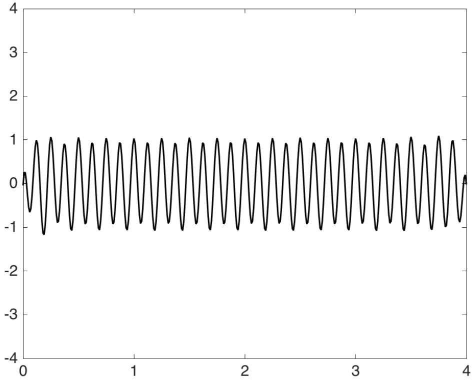

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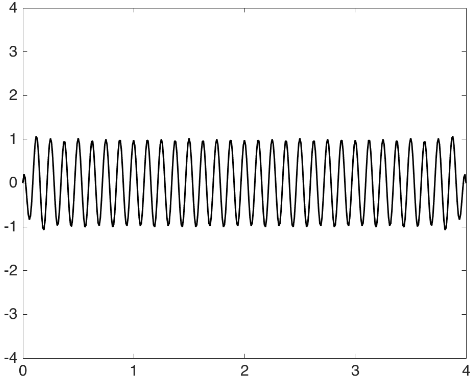

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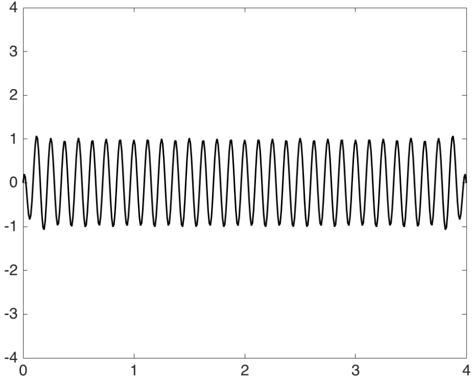

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  - symmetric about the time axis
  - # local extrema – # zeros = -1,0,1

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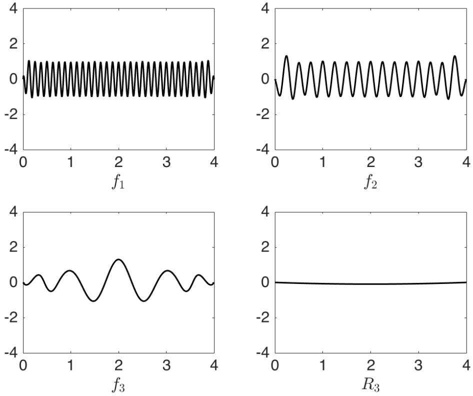

$$f(t) = \sum_{j=1}^{3} f_j(t) + R_3(t)$$

Hilbert spectral analysis

- We note that each IMF f_j can be written as

$$f_j(t) = \mathrm{Re}\left(f_j^{\star}(t)\right) = \mathrm{Re}\left(f_j(t) + i(\mathcal{H}f_j)(t)\right)$$

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Hilbert transform:

$$(\mathcal{H}g)(t) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(u)}{t-u}\,\mathrm{d}u, \quad g \text{ measurable, real-valued}$$

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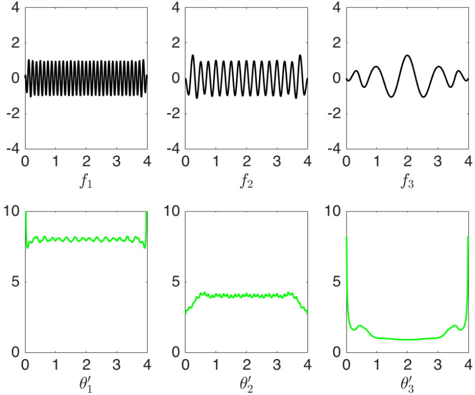
with

$$B_j(t) = |f_j^\star(t)| \text{ and } \theta_j(t) = \frac{1}{2\pi}\tan^{-1}\frac{\mathcal{H}f_j(t)}{f_j(t)} \text{ and } \mathrm{IF} = \theta_j'(t).$$

Final result:



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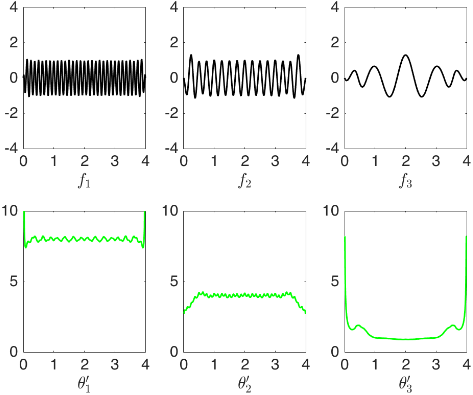

$$f(t) = \cos 2\pi(8t) + \cos 2\pi(4t) + \cos 2\pi t$$

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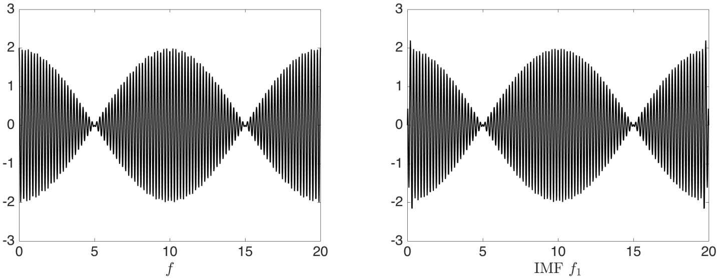

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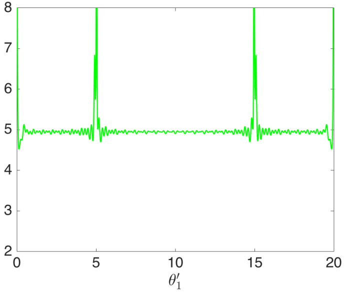

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- The sifting process cannot distinguish between atoms with frequencies that are very close together. In other words, a single IMF may actually contain more than one atom.
- We can get inaccurate results when computing the Hilbert transform from discrete samples on a bounded interval.
- The standard cubic spline interpolation scheme used to construct the upper and lower envelopes in the sifting process is not a local method.

Our contribution

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Given a function f, we construct an interpolation operator $\mathcal{P}$ in terms of cubic B-splines $N_{\mathbf{t},4,j}$ ...

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- non-decreasing knot sequence $\mathbf{t}$
- m^{th} order B-splines:

$$N_{\mathbf{t},1,j}(t) := \begin{cases} 1, & \text{if } t_j \leq t < t_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{\mathbf{t},m,j}(t) := w_{m,j} N_{\mathbf{t},m-1,j}(t) + (1 - w_{m,j+1}) N_{\mathbf{t},m-1,j+1}(t)$$

$$\text{with} \quad w_{m,j}(t) := \begin{cases} \frac{t-t_j}{t_{j+m-1}-t_j} & \text{if } t_j \neq t_{j+m-1} \\ 0, & \text{otherwise} \end{cases}$$

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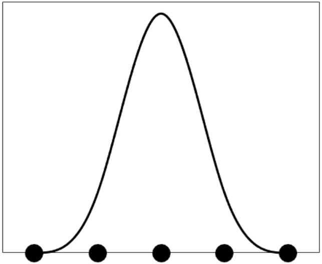

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Blending operator [Chui, Diamond, 1990]:

$$\mathcal{P} := \mathcal{Q} + \mathcal{R} - \mathcal{R}\mathcal{Q}$$

Quasi-interpolant $(\mathcal{Q}f)(t) :=$

$$\sum_{j=-3}^{-1} f^{(-j)}(a)M_j(t) + \sum_{j=0}^{n} f(t_j)M_j(t) + \sum_{j=n+1}^{n+2} f^{(j-n)}(b)M_j(t)$$

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Spline molecules [Chen, Chui, Lai, 1988; Chui, vdW, 2015]:

$$M_j(t) = \begin{cases} \displaystyle\sum_{k=0}^{j+3} a_{j,k} N_{\mathbf{t},4,k-3}(t), & j = -3, \dots, -1; \\ \displaystyle\sum_{k=0}^{3} a_{j,k} N_{\mathbf{t},4,j+k-3}(t), & j = 0, \dots, n-1; \\ \displaystyle\sum_{k=j-n}^{2} a_{j,k} N_{\mathbf{t},4,n+k-3}(t), & j = n, \dots, n+2. \end{cases}$$

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Find spline coefficients $a_{j,k}$ such that $(\mathcal{Q}p)(t) = p(t),\ p \in \pi_3$.

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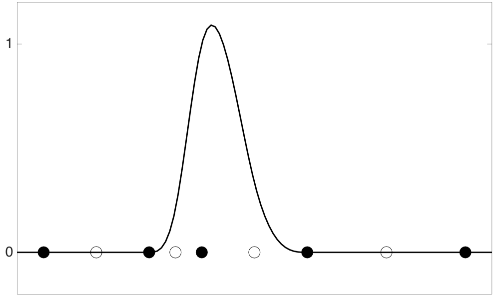

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$$L_j(t) = \begin{cases} \displaystyle\sum_{k=0}^{3} b_{j,k} N_{\tilde{\mathbf{t}}_{k-3},4}(t), \quad j=-3,\ldots,0; & \rightarrow L_j^{(\ell)}(a) = \delta_{-j,\ell} \\ \dfrac{N_{\tilde{\mathbf{t}},4,2j}(t)}{N_{\tilde{\mathbf{t}},4,2j}(t_j)}, \quad j=1,\ldots,n-1; & \rightarrow L_j(t_\ell) = \delta_{j,\ell} \\ \displaystyle\sum_{k=0}^{2} b_{j,k} N_{\tilde{\mathbf{t}}_{n+k},4}(t), \quad j=n,\ldots,n+2. & \rightarrow L_j^{(\ell)}(b) = \delta_{j-n,\ell} \end{cases}$$

Easy to show: $(\mathcal{R}f)(t_\ell) = f(t_\ell)$ and $\mathcal{R}$ preserves derivatives at a, b.

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$$(\mathcal{P}f)^{(j)}(a) = f^{(j)}(a),\ j = 1, 2, 3;$$

$$(\mathcal{P}f)^{(j)}(b) = f^{(j)}(b),\ j = 1, 2.$$

Limitations of EMD with Hilbert spectral analysis:

- The sifting process cannot distinguish between atoms with frequencies that are very close together. In other words, a single IMF may actually contain more than one atom.
- We can get inaccurate results when computing the Hilbert transform from discrete samples on a bounded interval.
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- EMD + SSO = "SuperEMD"

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Assumptions:

- $A_j \in C(\mathbb{R})$, $A_j(t) > 0$ and $\phi_j \in C^1(\mathbb{R})$, $\phi_j'(t) > 0$
- There exists $\alpha = \alpha(t) > 0$ s.t. for any u with $|u| \leq \alpha^{-1}(8\pi B)^{-1/2}$, where $B = B(t) := \max_j \phi_j'(t)$:
  - $|A_j(t+u) - A_j(t)| \leq \alpha^3 |u| A_j(t)$
  - $|\phi_j'(t+u) - \phi_j'(t)| \leq \alpha^3 |u| \phi_j'(t)$
- There exists $\eta = \eta(t)$ s.t. $\min_{j \neq k} |\phi_j'(t) - \phi_k'(t)| =: \frac{2B\eta}{\pi} > 0$

Note:

$$M = M(t) := \sum_{j=1}^{N} A_j(t) \quad \text{and} \quad \mu = \mu(t) := \min_{1 \leq j \leq N} A_j(t)$$

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$$(\mathcal{T}_{a,\delta} f_j)(t,\theta) = \frac{1}{\sum_{k\in\mathbb{Z}} h\left(\frac{k}{a}\right)} \sum_{\ell\in\mathbb{Z}} h\left(\frac{\ell}{a}\right) e^{i\ell\theta} f_j(t-\ell\delta)$$

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- It can be shown [Chui, Mhaskar 2015] that

$$\left\{\theta \in [0,\pi] : \left|\left(\mathcal{T}_{a,\delta} f_j\right)(t,\theta)\right| \geq \frac{\mu}{2}\right\}$$

consists of disjoint clusters $G_{j,1}, \ldots, G_{j,N_j}$, centered around $2\pi\delta\phi'_{j,1}(t), \ldots, 2\pi\delta\phi'_{j,N_j}(t)$.

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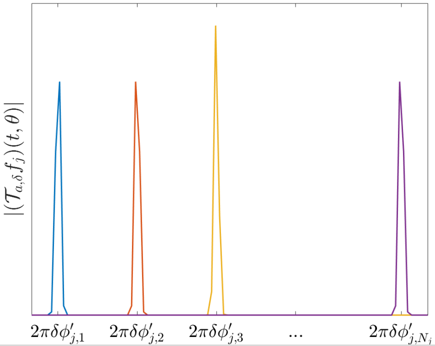

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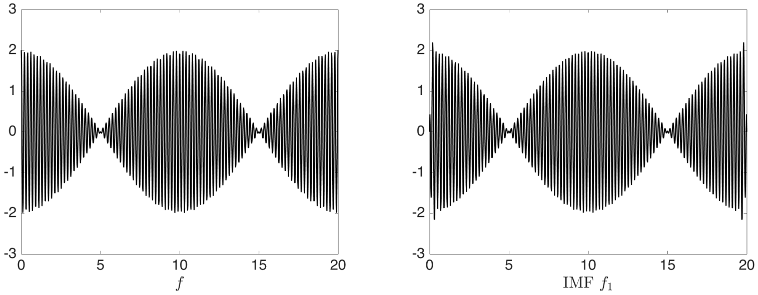
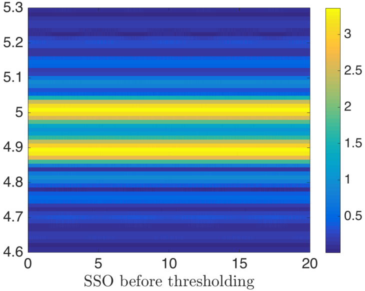
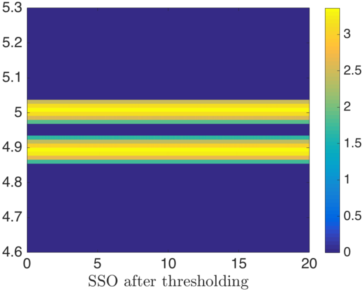
$\Big(0 < \delta \leq (4B)^{-1}$ and $a = (\alpha\delta\sqrt{8\pi B})^{-1}$ for sufficiently small $\alpha\Big)$



$$f(t) = \cos 2\pi(5t) + \cos 2\pi(4.9t)$$





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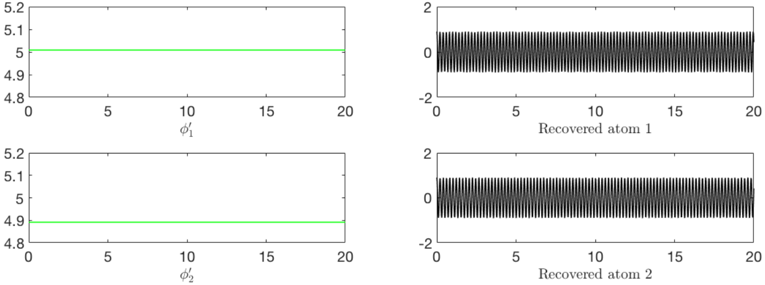

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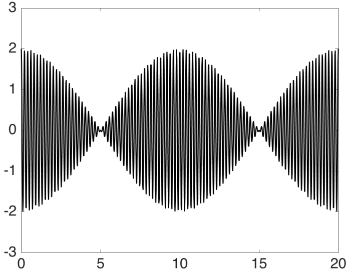
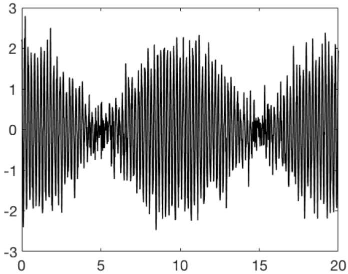


f without noise



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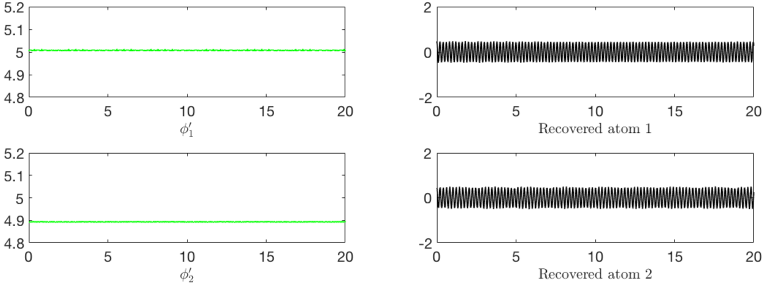

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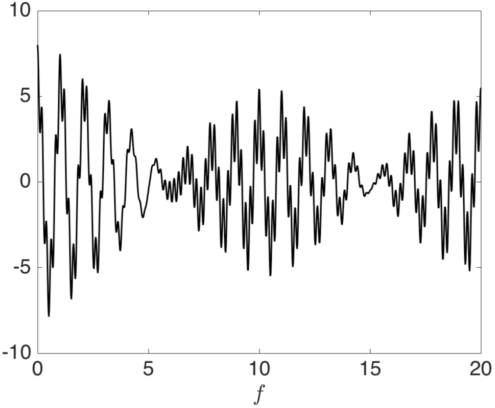

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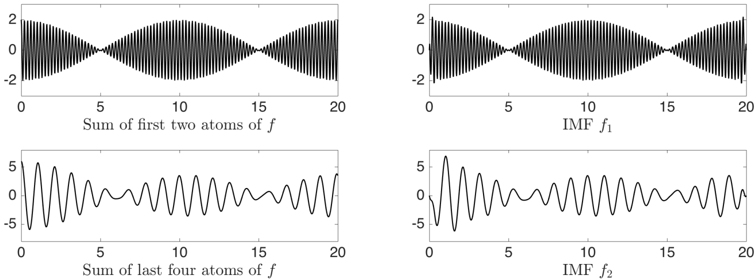

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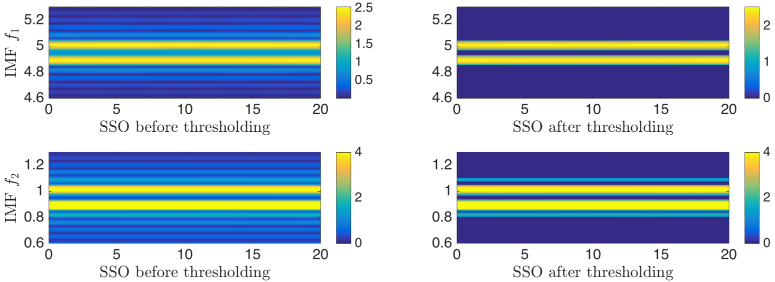

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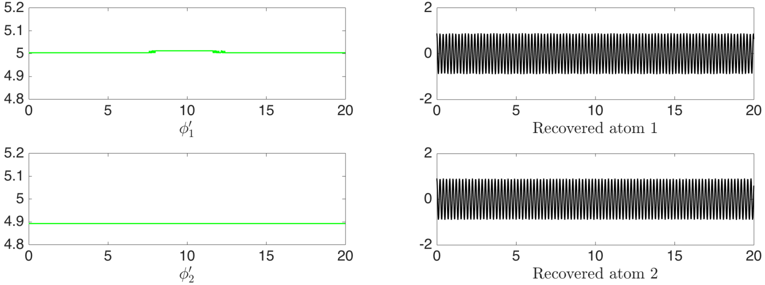

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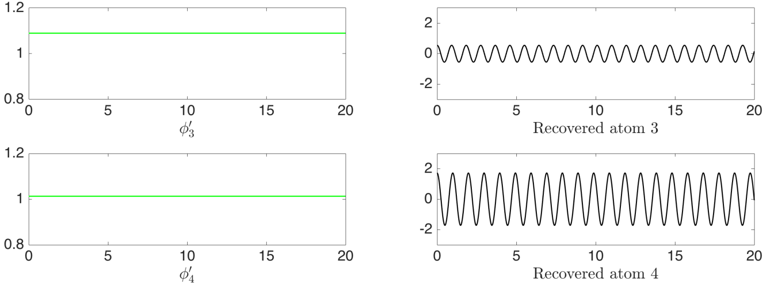

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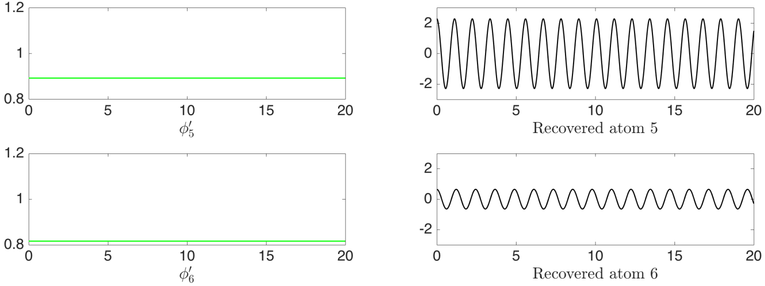

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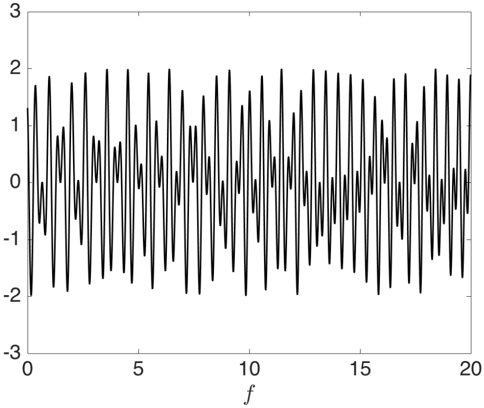

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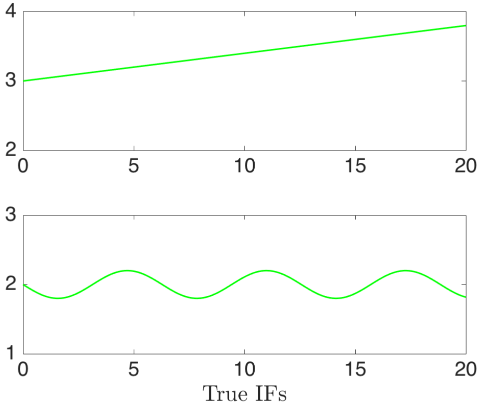

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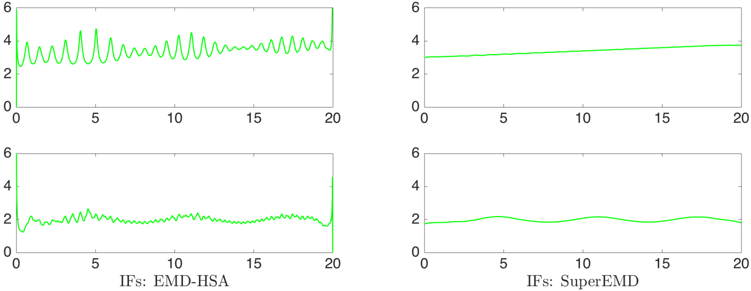

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Thank you for your attention.