Two Weight Inequalities for Calderón-Zygmund Operators

Brett D. Wick

Washington University in St. Louis
Department of Mathematics

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- A singular integral is an integral operator

$$T(f)(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\, dy$$

where the kernel function $K(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is singular along the diagonal $x = y$.

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where the kernel function $K(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is singular along the diagonal $x = y$.

- Typical conditions on the kernel are:

$$|K(x,y)| \leq \frac{C}{|x-y|^d} \quad \text{and} \quad |\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}}.$$

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  Plays an important role in complex analysis and harmonic analysis.
- Riesz transforms are the d-dimensional analogue of the Hilbert transform. Strong connection to the Laplacian. For $1 \leq j \leq d$ denote the Riesz transform in the jth variable by

$$R_j(f)(x) = \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

Formulation and Set up of the Problem

- For $0 \leq \lambda < d$ we define a smooth λ-fractional Calderón-Zygmund kernel $K^{\lambda}(x,y)$ to be a function $K^{\lambda} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following fractional size and smoothness conditions

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Question (Motivating Question)

Find necessary and sufficient conditions on a pair of weights σ and ω and a smooth λ-fractional singular integral operator T^λ on $\mathbb{R}^d$ to characterize when it satisfies the following two weight norm inequality

$$\left\|T^\lambda(\sigma f)\right\|_{L^p\left(\mathbb{R}^d,\omega\right)} \leq \mathfrak{N}_{T^\lambda,p}\left(\sigma,\omega\right) \left\|f\right\|_{L^p\left(\mathbb{R}^d,\sigma\right)}.$$

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- Dyadic (discrete) variants studied by NTV, Hytönen, Vuorinen, and others.

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Question (Carleson Measure Problem for K_ϑ)

Geometrically/function theoretically characterize the Carleson measures for K_ϑ:

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta.$$

Carleson Measures for K_ϑ

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- Nazarov and Volberg proved the obvious necessary condition is not sufficient for μ to be a K_ϑ-Carleson measure.

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Connecting Carleson Measures to the Cauchy Transform

- Associate to ϑ and $\alpha \in \mathbb{T}$ a measure σ_α such that:

$$\mathrm{Re}\left(\frac{\alpha+\vartheta(z)}{\alpha-\vartheta(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-\bar{\xi}z\right|^2}\, d\sigma_\alpha(\xi).$$

Let $\sigma \equiv \sigma_1$ denote the Clark measure on $\mathbb{T}$.

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Theorem (Nazarov, Volberg, (2002))

A measure μ is a Carleson measure for K_ϑ if and only if $\mathsf{C}: L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded.

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Observation

$$\left\|T^{\lambda}(\sigma f)\right\|_{L^p(\omega)} \leq C\left\|f\right\|_{L^p(\sigma)} \Leftrightarrow \left\|T^{\lambda,*}(\omega g)\right\|_{L^{p'}(\sigma)} \leq C\left\|g\right\|_{L^{p'}(\omega)}.$$

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&= \left|\int_{\mathbb{R}^d} f(y) T^{\lambda,*}(\omega g)(y)\, d\sigma(y)\right| = \left|\left\langle f, T^{\lambda,*}(\omega g)\right\rangle_{\sigma}\right|.
\end{aligned}$$

Invariance Under Adjoints

Observation

$$\left\|T^{\lambda}(\sigma f)\right\|_{L^p(\omega)} \leq C\|f\|_{L^p(\sigma)} \Leftrightarrow \left\|T^{\lambda,*}(\omega g)\right\|_{L^{p'}(\sigma)} \leq C\|g\|_{L^{p'}(\omega)}.$$

- Left hand side equivalent to:

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- Argument is reversible by interchanging the roles of σ and ω and f and g.

Simple Necessary Conditions

- Local scalar testing conditions:

$$\begin{aligned}\left\|\mathbf{1}_I T^{\lambda}(\sigma\mathbf{1}_I)\right\|_{L^p(\omega)} &\leq \mathfrak{T}_{T^{\lambda},p}(\sigma,\omega)\,|I|_{\sigma}^{\frac{1}{p}}\,,\\ \left\|\mathbf{1}_I T^{\lambda,*}(\omega\mathbf{1}_I)\right\|_{L^{p'}(\sigma)} &\leq \mathfrak{T}_{T^{\lambda,*},p'}(\omega,\sigma)\,|I|_{\omega}^{\frac{1}{p'}}\,.\end{aligned}$$

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- Weak boundedness property: For I and $J(I)$, any cube adjacent to I with the same length,

$$\left| \int_{\mathbb{R}^d} T^{\lambda}(\sigma \mathbf{1}_I)(x) \, \mathbf{1}_{J(I)}(x) \, d\omega(x) \right| \leq \mathcal{WBP}_{T^{\lambda},p}(\sigma,\omega) \, |I|_{\sigma}^{\frac{1}{p}} \, |J(I)|_{\omega}^{\frac{1}{p'}} .$$

Vector-Valued Extensions

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where the final equalities follow since T is linear. We have
$$\begin{aligned}\int_{\mathbb{R}^d} |T_{\mathbf{u}}\mathbf{f}(x)|^p \, d\omega(x) &= \int_{\mathbb{R}^d} |T\mathbf{f_u}(x)|^p \, d\omega(x) \\ &\leq \|T\|_{L^p(\sigma)\to L^p(\omega)}^p \int_{\mathbb{R}^d} |\mathbf{f_u}(x)|^p \, d\sigma(x) .\end{aligned}$$

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- Observe for a vector-valued function $\mathbf{F}(x)$ that:

$$\langle \mathbf{F}(x), \mathbf{u} \rangle = |\mathbf{F}(x)|_{\ell^2} \left\langle \frac{\mathbf{F}(x)}{|\mathbf{F}(x)|_{\ell^2}}, \mathbf{u} \right\rangle.$$

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$$\begin{aligned}
&\int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^d} |\langle \mathbf{F}(x), \mathbf{u} \rangle|^p \, d\mu(x) \right\} d\mathbf{u} \\
&= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{S}^{M-1}} |\langle \mathbf{F}(x), \mathbf{u} \rangle|^p \, d\mathbf{u} \right\} d\mu(x) \\
&= \int_{\mathbb{R}^d} |\mathbf{F}(x)|_{\ell^2}^p \left\{ \int_{\mathbb{S}^{M-1}} \left| \left\langle \frac{\mathbf{F}(x)}{|\mathbf{F}(x)|_{\ell^2}}, \mathbf{u} \right\rangle \right|^p d\mathbf{u} \right\} d\mu(x) \\
&= \gamma_p \int_{\mathbb{R}^n} |\mathbf{F}(x)|_{\ell^2}^p \, d\mu(x).
\end{aligned}$$

Vector-Valued Extensions

- Altogether then,

$$
\begin{aligned}
\gamma_p \int_{\mathbb{R}^d} |T\mathbf{f}(x)|_{\ell^2}^p \, d\omega(x) &= \int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^d} |T_{\mathbf{u}}\mathbf{f}(x)|^p \, d\omega(x) \right\} d\mathbf{u} \\
&= \int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^d} |T\mathbf{f}_{\mathbf{u}}(x)|^p \, d\omega(x) \right\} d\mathbf{u} \\
&\leq \int_{\mathbb{S}^{M-1}} \left\{ \|T\|_{L^p(\sigma)\to L^p(\omega)}^p \int_{\mathbb{R}^d} |\mathbf{f}_{\mathbf{u}}(x)|^p \, d\sigma(x) \right\} d\mathbf{u} \\
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- Dividing both sides by γ_p we conclude that

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- Let $M \nearrow \infty$ to obtain the ℓ^2vector-valued extension.

Equivalent Problem

Question (Motivating Question)

Find necessary and sufficient conditions on a pair of weights σ and ω and a smooth λ-fractional singular integral operator T^λ on $\mathbb{R}^d$ to characterize when it satisfies the following two weight norm inequality:

$$\left\|T^\lambda(\sigma f)\right\|_{L^p\left(\mathbb{R}^d,\omega\right)} \leq \mathfrak{N}_{T^\lambda,p}\left(\sigma,\omega\right)\|f\|_{L^p\left(\mathbb{R}^d,\sigma\right)}.$$

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Quadratic Weak Boundedness

- Quadratic Weak Boundedness:

$$\sum_{i=1}^{\infty} \sum_{I_i^* \in \mathrm{Adj}(I_i)} \left| \int_{\mathbb{R}^d} a_i T^{\lambda}(\sigma \mathbf{1}_{I_i})(x)\, b_i^* \mathbf{1}_{I_i^*}(x)\, d\omega(x) \right|$$

$$\leq \mathcal{WBP}_{T^{\lambda},p}^{\ell^2}(\sigma,\omega) \left\| \left(\sum_{i=1}^{\infty} \left| a_i \mathbf{1}_{I_i} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}$$

$$\times \left\| \left(\sum_{i=1}^{\infty} \sum_{I_i^* \in \mathrm{Adj}(I_i)} \left| b_i^* \mathbf{1}_{I_i^*} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\omega)},$$

where for $I \in \mathcal{D}$, its *adjacent* cubes are defined by

$$\mathrm{Adj}(I) \equiv \left\{ I^* \in \mathcal{D} : I^* \cap I \neq \emptyset \text{ and } \ell(I^*) = \ell(I) \right\}.$$

Quadratic Testing Conditions

- The *local quadratic* cube testing conditions are

$$\left\| \left(\sum_{i=1}^{\infty} \left| a_i \mathbf{1}_{I_i} T^{\lambda}(\sigma \mathbf{1}_{I_i}) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \leq \mathfrak{T}^{\mathrm{quad}}_{T^{\lambda},p}(\sigma,\omega) \left\| \left(\sum_{i=1}^{\infty} \left| a_i \mathbf{1}_{I_i} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)},$$

$$\left\| \left(\sum_{i=1}^{\infty} \left| a_i \mathbf{1}_{I_i} T^{\lambda,*}(\omega \mathbf{1}_{I_i}) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\sigma)} \leq \mathfrak{T}^{\mathrm{quad}}_{T^{\lambda,*},p'}(\omega,\sigma) \left\| \left(\sum_{i=1}^{\infty} \left| a_i \mathbf{1}_{I_i} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\omega)}$$

taken over all sequences $\{I_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=1}^{\infty}$ of cubes and numbers respectively.

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  taken over all sequences $\{I_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=1}^{\infty}$ of cubes and numbers respectively.
- The corresponding quadratic *global* cube testing constants $\mathfrak{T}^{\text{quad,global}}_{T^{\lambda},p}(\sigma,\omega)$ and $\mathfrak{T}^{\text{quad,global}}_{T^{\lambda,*},p'}(\omega,\sigma)$ are defined as above, but *without* the indicator $\mathbf{1}_{I_i}$ outside the operator, namely with $\mathbf{1}_{I_i} T^{\lambda}(\sigma \mathbf{1}_{I_i})$ replaced by $T^{\lambda}(\sigma \mathbf{1}_{I_i})$ and symmetrically.

Muckenhoupt Conditions

- T^λ is Stein elliptic if there is a choice of constant C and appropriate cubes I^* such that

$$\left|T^\lambda(\sigma\mathbf{1}_I)\,(x)\right| \geq c\frac{|I|_\sigma}{|I|^{1-\frac{\lambda}{n}}} \text{ for } x \in I^*.$$

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- Necessity of the quadratic offset $A_p^{\lambda,\ell^2,\mathrm{offset}}$ condition then follows from the global quadratic testing condition. The condition is:

$$\left\| \left(\sum_{i=1}^{\infty} \left(a_i \mathbf{1}_{I_i^*} \frac{|I_i|_\sigma}{|I_i|^{1-\frac{\lambda}{n}}} \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \leq A_p^{\lambda,\ell^2,\mathrm{offset}}(\sigma,\omega) \left\| \left(\sum_{i=1}^{\infty} |a_i \mathbf{1}_{I_i}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}$$

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taken over all sequences $\{I_i\}_{i=1}^\infty$ and $\{a_i\}_{i=1}^\infty$ of cubes and constants respectively. Dual versions by interchanging σ and ω and p and p'.

- Scalar versions exist: $\frac{|I|_\sigma^{\frac{1}{p'}}|I|_\omega^{\frac{1}{p}}}{|I|^{1-\frac{\lambda}{n}}} \leq A_p^\lambda(\sigma,\omega)$.

Main Theorem in the Doubling Setting

Theorem (E. Sawyer and B. D. Wick, (2022))

Suppose that $1 < p < \infty$, that σ and ω are locally finite positive Borel measures on $\mathbb{R}^d$. Then

$$\mathfrak{T}_{T^\lambda,p}(\sigma,\omega) + \mathfrak{T}_{T^{\lambda,*},p'}(\omega,\sigma) + \mathcal{WBP}^{\ell^2}_{T^\lambda,p}(\sigma,\omega) \lesssim \mathfrak{N}_{T^\lambda,p}(\sigma,\omega)$$

and when T^λ is Stein elliptic, we also have

$$A_p^{\lambda,\ell^2,\mathrm{offset}}(\sigma,\omega) + A_{p'}^{\lambda,\ell^2,\mathrm{offset}}(\omega,\sigma) \lesssim \mathfrak{N}_{T^\lambda,p}(\sigma,\omega).$$

If additionally, σ and ω are doubling measures on $\mathbb{R}^d$. Then

$$\begin{aligned}\mathfrak{N}_{T^\lambda,p}(\sigma,\omega) \quad &\lesssim \quad \mathfrak{T}_{T^\lambda,p}(\sigma,\omega) + \mathfrak{T}_{T^{\lambda,*},p'}(\omega,\sigma) + \mathcal{WBP}^{\ell^2}_{T^\lambda,p}(\sigma,\omega) \\ &\quad + A_p^{\lambda,\ell^2,\mathrm{offset}}(\sigma,\omega) + A_{p'}^{\lambda,\ell^2,\mathrm{offset}}(\omega,\sigma).\end{aligned}$$

Conjecture of Hytönen and Vuorinen

Conjecture (Hytönen and Vuorinen)

Suppose $1 < p < \infty$ *and that* σ *and* ω *are locally finite positive Borel measures on* $\mathbb{R}$*. Then the two weight norm inequality for the Hilbert transform holds if and only if the global quadratic interval testing conditions hold. Moreover, we have the equivalence*

$$\mathfrak{N}_{H,p}(\sigma,\omega) \approx \mathfrak{T}_{H,p}^{\ell^2,\mathrm{glob}}(\sigma,\omega) + \mathfrak{T}_{H,p'}^{\ell^2,\mathrm{glob}}(\omega,\sigma).$$

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Partial Progress on the Hytönen-Vuorinen Conjecture

Theorem (E. Sawyer and B. D. Wick (2023))

Suppose $p \in \left(\frac{4}{3}, 4\right)$ and that σ and ω are locally finite positive Borel measures on $\mathbb{R}$ without common point masses. Then the two weight norm inequality for the Hilbert transform holds if and only if the local quadratic interval testing conditions hold, the global Muckenhoupt condition holds, and the quadratic weak boundedness property holds. Moreover, we have the equivalence

$$\begin{aligned}\mathfrak{N}_{H,p}(\sigma,\omega) \quad &\approx \quad \mathfrak{T}^{\ell^2,\mathrm{loc}}_{H,p}(\sigma,\omega) + \mathfrak{T}^{\ell^2,\mathrm{loc}}_{H,p'}(\omega,\sigma) + \mathcal{WBP}^{\ell^2}_{H,p}(\sigma,\omega) \\ &\quad + \mathcal{A}^{\ell^2,\mathrm{glob}}_{p}(\sigma,\omega) + \mathcal{A}^{\ell^2,\mathrm{glob}}_{p'}(\omega,\sigma).\end{aligned}$$

Sobolev Space in the Weighted Setting

Definition

Let μ be a doubling measure on $\mathbb{R}^n$. Given $s \in \mathbb{R}$, we define the $\mathcal{D}$-dyadic homogeneous $W_{\mathcal{D}}^{s}(\mu)$-Sobolev norm of a function $f \in L_{\mathrm{loc}}^{2}(\mu)$ by

$$\|f\|_{W_{\mathcal{D}}^{s}(\mu)}^{2} \equiv \sum_{Q \in \mathcal{D}} \ell(Q)^{-2s} \left\| \triangle_{Q}^{\mu} f \right\|_{L^{2}(\mu)}^{2} ,$$

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Definition

For $s > 0$ and small enough and μ doubling, there is a familiar 'continuous' norm,

$$\|f\|_{W^s(\mu)} = \sqrt{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{f(x) - f(y)}{|x-y|^s} \right)^2 \frac{d\mu(x)\, d\mu(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2} \right) \right|_{\mu}}}.$$

Sobolev Version in the L^2 Setting

Theorem (E. Sawyer and B. D. Wick, (Math Z. 2022))

Let σ and ω be doubling Borel measures on $\mathbb{R}^d$. Then if $0 < s < \theta$, with θ depending on the doubling constants of σ and ω it holds

$$\left\| T^{\lambda}(\sigma f) \right\|_{W^s(\omega)} \lesssim \left(A_2^{\lambda} + \mathfrak{T}_{T^{\lambda}} + \mathfrak{T}_{T^{\lambda,*}} \right) \|f\|_{W^s(\sigma)},$$

provided the fractional Muckenhoupt condition and the Sobolev testing conditions are finite, where

$$\begin{aligned} A_2^{\lambda} &\equiv \sup_{Q \in \mathcal{Q}^n} \frac{|Q|_{\omega} |Q|_{\sigma}}{|Q|^{2\left(1-\frac{\lambda}{n}\right)}} \\ \left\| T_{\sigma}^{\lambda} \mathbf{1}_I \right\|_{W^s(\omega)} &\leq \mathfrak{T}_{T^{\lambda}}(\sigma, \omega) \sqrt{|I|_{\sigma}} \ell(I)^{-s}, \qquad I \in \mathcal{Q}^n, \\ \left\| T_{\omega}^{\lambda,*} \mathbf{1}_I \right\|_{W^{-s}(\sigma)} &\leq \mathfrak{T}_{T^{\lambda,*}}(\omega, \sigma) \sqrt{|I|_{\omega}} \ell(I)^{s}, \qquad I \in \mathcal{Q}^n. \end{aligned}$$

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- Key Fact: For $1 < p < \infty$,

$$\left\|\mathcal{S}^\mu f\right\|_{L^p(\mu)} \approx \left\|f\right\|_{L^p(\mu)}.$$

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- Expand the bilinear form associated to T^λ:

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- Typical example can be written as:

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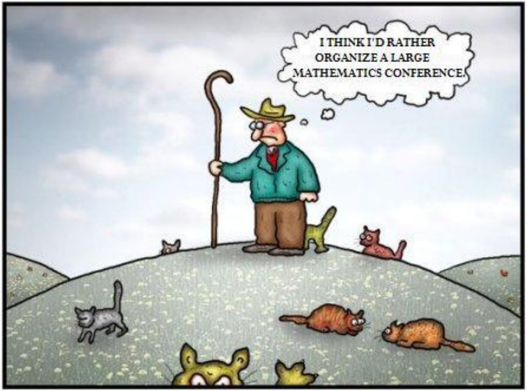
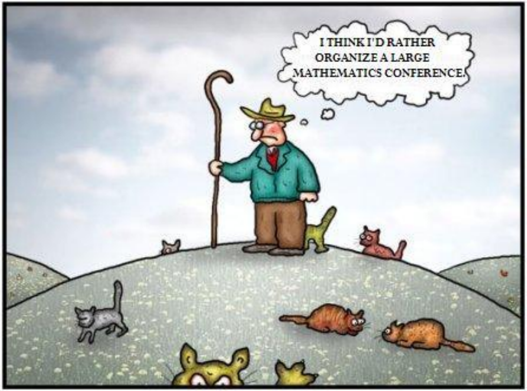
  - Square function estimates, and variants, are used to control the factor involving $\|g\|_{L^{p'}(\omega)}$.



The daydreams of cat herders



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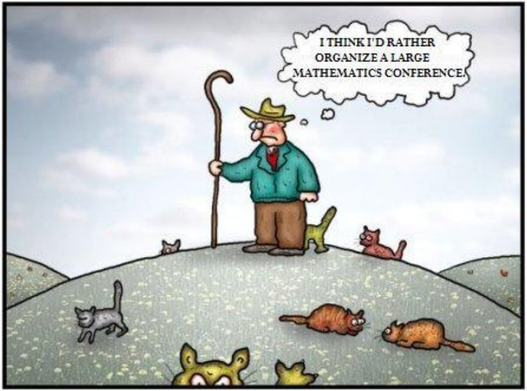

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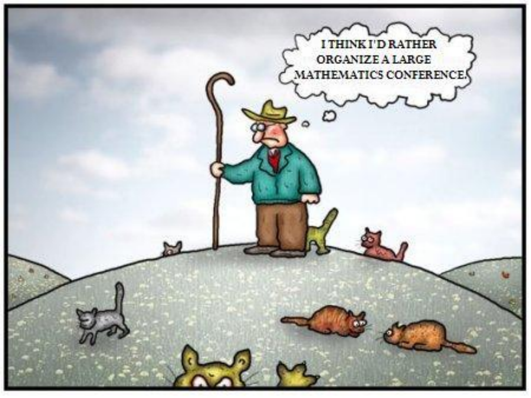

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Thanks to the Organizers for Arranging MWAA!