Embeddability of real hyersurfaces into hyperquadrics and spheres

Ming Xiao
University of California San Diego

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Question

When a real hypersurface $M \subset \mathbb{C}^n$ admits a holomorphic transversal embedding into a hyperquadric $\mathbb{H}_l^{2N-1} \subset \mathbb{C}^N$ of possibly larger dimension?

Preliminary

- **Hyperquadrics:**

$$\mathbb{H}_l^{2N-1} := \{-\sum_{i=1}^{l} |z_i|^2 + \sum_{i=l+1}^{N} |z_i|^2 = 1\} \subset \mathbb{C}^N.$$

- **Transversal map** F :

dF does not map $T_p\mathbb{C}^n$ to $T_{F(p)}\mathbb{H}_l^{2N-1}$ at $p \in M$.

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- Differential Geometry: The Whitney embedding theorem

  Embedding of general smooth manifolds into their models (real Euclidean spaces)

- Riemannian Geometry: The Nash embedding theorem

  Embedding of general Riemannian manifolds into their models (real Euclidean spaces)

Preliminary

- Stein Space theory: The Remmert embedding theorem

  Embedding of Stein manifolds into their models (complex Euclidean spaces)

- Pseudoconformal geometry: One may ask whether there is such an analogue.

  Embedding of hypersurfaces into their models (hyperquadrics)

Preliminary

Webster, 1978

Theorem

(Webster, 1978, Duke Math. J.) Every real-algebraic Levi-nondegenerate real hypersurface $M \subset \mathbb{C}^n$ is transversally holomorphically embeddable into a hyperquadric of suitable dimension and signature.

Preliminary

However, not every real analytic Levi-nondegenerate hypersurface can be transversally holomorphically embedded into a hyperquadric of sufficient large dimension.

Preliminary

- Forstnerić 1986, Faran 1988
  Most real analytic strongly pseudoconvex hypersurface cannot be holomorphically embedded into any sphere.

- Forstnerć 2004
  Most real-analytic hypersurfaces do not admit a transversal holomorphic embedding into any real algebraic hypersurface, in particular, any hyperquadrics.

- Explicit Example:

Theorem

(Zaitsev, 2008, Math. Ann.)The hypersurface in $\mathbb{C}^2$ given by

$$\mathrm{Im} w = |z|^2 + \mathrm{Re} \sum_{k \geq 2} z^k \overline{z}^{(k+2)!}, (z, w) \in \mathbb{C}^2, |z| < \epsilon.$$

for any $0 < \epsilon \leq 1$ is not transversally holomorphically embeddable into a hyperquadric of any dimension.

Motivated by Webster's theorem and embedding theorems in geometry:

Equivalently,

Is there a uniform bound for the minimal embedding dimension of M in terms of n :

Theorem

(Kossovskiy-X., to appear in Advances in Math.) For any integers $N > n > 1$, *there exists* $\mu = \mu(n, N)$ *such tha a Zariski generic real-algebraic hypersurface* $M \subset \mathbb{C}^n$ *of degree* $k \geq \mu$ *is not transversally holomorphically embeddable into any hyperquadric* $\mathbb{H}_l^{2N-1} \subset \mathbb{C}^N$.

Main Results

- We can give an explicit bound for $\mu(n, N)$:

$$\mu(n, N) = 2 + N - n + \begin{pmatrix} N(N+1)/2 + p(n, N) \\ p(n, N) \end{pmatrix},$$

where $p(n, N) = n - 1 + \frac{(n-1)n}{2} \begin{pmatrix} N - 1 \\ n - 1 \end{pmatrix}$.

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- When $n = 2, N = 3$, we have $\mu(2, 3) = 18$.

We now concentrate on the strongly pseudoconvex case:

Question

Is every compact real-algebraic strongly pseudoconvex real hypersuraface in $\mathbb{C}^n$ holomorphically embeddable into a sphere of sufficiently large dimension?

In Huang-Li-X., 2015, I.M.R.N, the hypersurfaces $M_\epsilon \subset \mathbb{C}^2$ are constructed:

$$M_\epsilon := \{(z, w) \in \mathbb{C}^2 : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \varepsilon|z|^2 - 1 = 0\},$$

where $0 < \varepsilon < 1, 0 < \varepsilon_0 << 1, 2 < c < \frac{16}{7}$.

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- M_ϵ is a real algebraic hypersurface.
- For small $\varepsilon, \varepsilon_0$, M_ε is diffeomorphic to $\mathbb{S}^3 \subset \mathbb{C}^2$.
- For $0 < \varepsilon < 1$, M_ε is strongly pseudoconvex.
- M_0 has a Kohn-Nirenberg point at $(0, 1)$.

Theorem

(Huang-X., 2016) For sufficient small $\varepsilon, \varepsilon_0,\ \ M_\varepsilon$ *cannot be locally holomorphically embedded into any sphere. More precisely, any holomorphic map that sends an open piece of* M_ε *to a unit sphere must be constant.*

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We thus give a negative answer to the question:

There exist compact, real algebraic, strongly pseudoconvex hypersurfaces that cannot be locally holomorphically embedded into any sphere.

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  Huang's algebraicity theorem.

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- Step 1 (c): A theorem of Chiappari $\Rightarrow$ F extends to a holomorphic map in a neighborhood of $\overline{D_\varepsilon}$.

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- Note p_0 is on M_ε for every $0 \leq \varepsilon < 1$. Moreover,

$$Q_{p_0} = \{w = 1\}.$$

Recall
$M_\varepsilon : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \varepsilon|z|^2 - 1 = 0.$

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- There is $\widetilde{p} \in Q_{p_0}$ such that $\widetilde{p} \in D_0$.
- For $0 < \varepsilon << 1$, $\widetilde{p} \in D_\varepsilon$.
- $Q_{p_0} \cap M_\varepsilon$ is of real dimension one $\Rightarrow$ $Q_{p_0} \cap M_\varepsilon$ has an accumulation point q_0.

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- By unique continuation, if $q \in Q_{p_0} \cap M_\varepsilon$, then

$$F(q) \in \widetilde{Q}_{F(p_0)} \cap \mathbb{S}^{2N-1} = \{F(p_0)\}.$$

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  This is a contradiction.

Remarks

- Each

$$M_\varepsilon : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \varepsilon|z|^2 - 1 = 0$$

can be transversally holomorphically embedded into the hyperquadric in $\mathbb{C}^6$ with one negative Levi eigenvalue:

$$\mathbb{H}_1^{11} = \left\{ (z_1, ..., z_6) \in \mathbb{C}^6 : \sum_{i=1}^{5} |z_i|^2 - |z_6|^2 = 1 \right\}$$

by

$$F(z, w) = \left(\sqrt{\varepsilon_0} z^4, z^5, \sqrt{\varepsilon} z, w, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 + z), \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 - z) \right).$$

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- A lot more examples can be constructed.

Thank you very much for your attention!