# Unique continuation for $\bar{\partial}$ with square-integrable potentials

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# A fact/An example:

Let $\Omega$ be a domain in $\mathbb{R}^d$. Consider the Laplace equation: $\Delta u = 0$ on $\Omega$.
$\Longrightarrow$ Any smooth solution $u$ satisfying

$$\partial_x^m u(x_0) = 0, \quad \text{for all} \quad m \in \mathbb{Z}^+ \cup \{0\}$$

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## A further fact:

Any weak solution $u \in H^2_{loc}(\Omega)$ to $\Delta u = 0$ on $\Omega$ satisfying

$$\lim_{r \to 0} r^{-m} \int_{|x - x_0| < r} |u(x)|^2 dv_x = 0, \tag{1}$$

at $x_0$ for all $m \in \mathbb{Z}^+ \cup \{0\}$ vanishes identically.

Definition: A function $u \in L^2_{loc}(\Omega)$ is said to *vanish to infinite order, or flat, in $L^2$ sense* at $x_0 \in \Omega$ if (1) holds.

# Unique continuation property

Let $\Omega$ be a domain in $\mathbb{R}^d$. Consider a differential equation

$$P_m(x, \partial_x)\, u = 0, \tag{2}$$

where $u = (u_1, \ldots, u_{d'}) : \Omega \to \mathbb{R}^{d'}$ with $u \in H^m_{loc}(\Omega)$.

## Definition

- (2) is said to satisfy the *strong unique continuation property* (strong UCP), if every solution that vanishes to infinite order in $L^2$ sense at a point $z_0 \in \Omega$ vanishes identically.
- (2) is said to satisfy the *weak unique continuation property* (weak UCP), if every solution that vanishes in an open subset vanishes identically.

**Remark:** strong UCP $\Longrightarrow$ weak UCP.

# Unique continuation for $\bar{\partial}$

## Question:

Let $\Omega$ be a domain in $\mathbb{C}^n$ and $V \in L^2_{loc}(\Omega)$. Consider the following inequality

$$|\bar{\partial} u| \leq V|u|, \tag{3}$$

where $u = (u_1, \ldots, u_N) : \Omega \to \mathbb{C}^N$ with $u \in H^1_{loc}(\Omega)$. Does UCP holds for (3)?

# Motivation

- Boundary behavior of holomorphic functions [Bell-Lempert, JDG, 1990]:
  $\Omega_1 \subset \mathbb{C}, \Omega_2 \subset \mathbb{C}^N$ with desired regularity and geometry;
  $f : \Omega_1 \to \Omega_2$ be proper holomorphic with certain regularity on $b\Omega_1$.
  $\implies$ $f$ is flat at $z_0 \in b\Omega_1$ iff UCP holds for

$$\bar{\partial} u = Vu,$$

  for some potential $V$ determined by $\Omega_1, \Omega_2$.
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  Bell-Lempert proved the strong UCP holds when $n = 1$ and $V \in L^\infty$.
- Almost complex structure:
  $(M, J)$: an almost complex manifold with desired regularity;
  $u_1, u_2 : D(\subset \mathbb{C}) \to M$ are $J$-holomorphic curves near $z_0 \in M$ (namely, $du_j$ commutes with $J$), and $u_1 - u_2$ is flat at $z_0$.
  $\implies u_1 \equiv u_2$ iff UCP holds for
  $$|\bar{\partial} u| \le V|u|,$$
  for some potential $V$ determined by almost complex structures.

# $L^2$ potential is critical

Assume $V \in L^p, p < 2$.

- The strong UCP fails in general.

**Example**

For each $0 < p < 2$, choose $\epsilon \in (0, \frac{2-p}{p})$ (so that $(\epsilon + 1)p < 2$). Let $V = \frac{\epsilon}{2|z|^{\epsilon+1}} \in L^p(D)$ on $\mathbb{C}$. $u_\epsilon(z) := e^{-\frac{1}{|z|^\epsilon}}$ vanishes to infinite order at 0 and satisfies $\bar{\partial} u = Vu$.

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- The weak UCP fails in general.

**Example (Mandache, Comm. Anal. Geom. 2002)**

There exist non-trivial $u \in C_c^\infty(\mathbb{C})$ and $V \in L^p(\mathbb{C}), 0 < p < 2$, such that $u$ satisfies $\bar{\partial} u = Vu$.

# UCP for Laplacian

**Theorem (Chanillo-Sawyer, Trans. Amer. Math. Soc. 1990 )**

*Let $\Omega$ be a domain in $\mathbb{R}^d$ with $d = 2$. Suppose $u = (u_1, \ldots, u_{d'}) : \Omega \to \mathbb{R}^{d'}$ with $u \in H^2_{loc}(\Omega)$ and satisfies*

$$|\Delta u| \leq V|\nabla u| \tag{4}$$

*for some $V \in L^2_{loc}(\Omega)$. If $u$ vanishes to infinite order at $z_0 \in \Omega$, then $u$ vanishes identically.*

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**Remark:** The result holds when $d = 3, 4$ and $V \in L^d_{loc}(\Omega)$ [Wolff, Rev Math. Iberoamericana, 1990].

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**Remark:** The result holds when $d = 3, 4$ and $V \in L^d_{loc}(\Omega)$ [Wolff, Rev Math. Iberoamericana, 1990].

**Example (Wolff, Comm. Anal. Geom. 1994)**

There exists a smooth real-valued function on $\mathbb{R}^d, d \geq 5$, which vanishes to infinite order at the origin and satisfies (4) with $V \in L^d(\mathbb{R}^d)$.

# Non-equivalence between $\Delta$ and $\bar{\partial}$ when $d = 2$

**Question:**

UCP holds for $|\bar{\partial} v| \leq V|v| \overset{???}{\Longleftrightarrow}$ UCP holds for $|\Delta u| \leq V|\nabla u|$.

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- UCP holds for $|\bar{\partial} v| \le V|v| \implies$ UCP holds for $|\Delta u| \le V|\nabla u|$.

  Use $\Delta = 4\bar{\partial}\partial$ and $|\partial u| = \frac{1}{2}|\nabla u|$ (since $u$ is real-valued). Then $v := \partial u$ is flat and satisfies $|\bar{\partial} v| \le V|v|$.

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  Argue reversely. However an obstruction appears: solve $u$ to $\partial u = v$ in the flat category.

# An obstruction in flat category

**Theorem (Liu-Pan-Z., Integral Equations Operator Theory, to appear)**

*Let $f$ be a flat germ at the origin in $\mathbb{C}$. The following two statements are equivalent:*
*1) The Cauchy-Riemann equation $\bar{\partial} u = f d\bar{z}$ has a solution that is flat at the origin in the sense of germs.*
*2) There exists some neighborhood $U \subset \mathbb{C}$ of the origin such that*

$$\sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_U \frac{f(\zeta)}{\zeta^{n+1}} d\bar{\zeta} \wedge d\zeta \right) z^n$$

*is holomorphic near the origin.*

# Main theorem

**Theorem (Pan-Z., preprint 2022)**

*Let $\Omega$ be a domain in $\mathbb{C}$. Suppose $u = (u_1, \ldots, u_N) : \Omega \to \mathbb{C}^N$ with $u \in H^1_{loc}(\Omega)$ and satisfies $|\bar{\partial} u| \leq V|u|$ for some $V \in L^2_{loc}(\Omega)$. If $u$ vanishes to infinite order at $z_0 \in \Omega$, then $u$ vanishes identically.*

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**Example**

For $0 < \epsilon < \frac{1}{2}$, let

$$V(z) = \frac{\epsilon}{2|z| \left(-\ln |z|\right)^{1-\epsilon}}$$

on $D_{\frac{1}{2}}$. Then $V \in L^2(D_{\frac{1}{2}})$. Then $|\bar{\partial} u| = V|u|$ has no nontrivial flat solution.

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- $u_0(z) = e^{-(-\ln|z|)^{\epsilon}} \in H^1(D_{\frac{1}{2}})$ and satisfies $|\bar{\partial} u| = V|u|$ on $D_{\frac{1}{2}}$. However, $u_0$ is not flat anywhere on $D_{\frac{1}{2}}$.
- $V \notin L^p(D_{\frac{1}{2}})$ for any $p > 2$.

# Weak UCP in $\mathbb{C}^n, n \geq 2$

## Theorem

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**Remark:** 1. The weak UCP fails for $|\bar{\partial} u| \leq V|u|$ if the potential $V$ is at most $L^p, p < 2$.

**Example (Mandache, Comm. Anal. Geom. 2002)**

For each $0 < p < 2$, there exist nontrivial $u \in C_c^\infty(\mathbb{C}^2), V \in L^p(\mathbb{C}^2)$ such that $u$ satisfies $|\bar{\partial} u| \leq V|u|$ on $\mathbb{C}^2$.

# Weak UCP in $\mathbb{C}^n, n \geq 2$

**Theorem**

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2. The weak UCP for $|\Delta u| \leq V|\nabla u|$ holds with $V \in L^{dim_{\mathbb{R}}\Omega}_{loc}$ [Wolff, Geom. Funct. Anal, 1992], and fails with $V \in L^p(\Omega), p < dim_{\mathbb{R}}\Omega$. [Koch-Tataru, Crelles Jour, 2002].

# Proof of the main theorem

**Lemma 1**

Let $u \in H^1(\mathbb{C})$ has compact support. Then for almost every $z \in \mathbb{C}$,

$$u(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial} u(\zeta)}{z - \zeta} dv_\zeta.$$

If in addition that $u$ vanishes near 0, then for any $m \geq 0$,

$$\int_{\mathbb{C}} \frac{\bar{\partial} u(\zeta)}{\zeta^{m+1}} dv_\zeta = 0.$$

# Proof of the main theorem, continued

The Riesz fractional integral of $f$ of order 1 is

$$I_1 f(z) := \int_{\mathbb{C}} \frac{f(\zeta)}{|\zeta - z|} dv_\zeta, \quad z \in \mathbb{C}.$$

$L^2_V(\Omega)$ with respect to a given $V(>0)$ on $\Omega \subset \mathbb{C}$ is the collection of all functions $f$ on $\Omega$ such that

$$\|f\|_{L^2_V(\Omega)} := \left( \int_\Omega |f(z)|^2 V(z) dv_z \right)^{\frac{1}{2}} < \infty.$$

**Lemma 2 (Weighted Hardy-Littlewood-Sobolev inequality)**

Let $V \in L^2(\mathbb{C})$. Then there exists a universal constant $C$ such that for any $f \in L^2_{V^{-1}}(\mathbb{C})$,

$$\|I_1 f\|_{L^2_V(\mathbb{C})} \leq C \|V\|_{L^2(\mathbb{C})} \|f\|_{L^2_{V^{-1}}(\mathbb{C})}.$$

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**Remark:** If $V \in L^p(\mathbb{C}), p > 2$, then by Hölder inequality,

$$\|I_1 f\|_{L^2_V(D_1)} \leq C_p \|V\|_{L^p(D_1)} \|f\|_{L^2_{V^{-1}}(D_1)} \text{ for all } f \in L^2_{V^{-1}}(D_1).$$

# Proof of the main theorem, continued

Assume $u$ is flat at $z = 0$.

- Choose $\eta \in C_c^\infty(\mathbb{C})$ such that $\eta = 1$ on $D_r$; $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{2}{r}$ on $D_{2r} \setminus D_r$; $\eta = 0$ outside $D_{2r}$.
- Let $\psi \in C^\infty(\mathbb{C})$ be such that $\psi = 0$ in $D_1$; $0 \le \psi \le 1$ and $|\nabla \psi| \le 2$ on $D_2 \setminus D_1$; $\psi = 1$ outside $D_2$. Let $\psi_k(z) = \psi(kz), z \in \mathbb{C}, k \ge \frac{2}{r}$.

$\implies \psi_k \eta u \in H^1(\mathbb{C})$ and is supported inside $D_{2r} \setminus D_{\frac{1}{k}}$.

# Proof of the main theorem, continued

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$\implies \psi_k \eta u \in H^1(\mathbb{C})$ and is supported inside $D_{2r} \setminus D_{\frac{1}{k}}$.

On $D_r$,

$$|\psi_k(z)|^2|u(z)|^2 = \frac{1}{\pi^2}\left|\int_{\mathbb{C}} \frac{\bar\partial(\psi_k(\zeta)\eta(\zeta)u(\zeta))}{z-\zeta} dv_\zeta\right|^2$$

$$= \frac{1}{\pi^2}\left|\int_{\mathbb{C}} \left(\frac{1}{z-\zeta} + \sum_{l=0}^{m-1} \frac{z^l}{\zeta^{l+1}}\right) \bar\partial\left(\psi_k(\zeta)\eta(\zeta)u(\zeta)\right) dv_\zeta\right|^2.$$

# Proof of the main theorem, continued

$$\int_{D_r} \frac{|\psi_k(z)|^2 |u(z)|^2}{|z|^{2m}} V(z) dv_z$$

$$= \frac{1}{\pi^2} \int_{D_r} \frac{1}{|z|^{2m}} \left| \int_{\mathbb{C}} \left( \frac{1}{z-\zeta} + \sum_{l=0}^{m-1} \frac{z^l}{\zeta^{l+1}} \right) \bar{\partial} \left( \psi_k(\zeta)\eta(\zeta)u(\zeta) \right) dv_\zeta \right|^2 V(z) dv_z.$$

Apply $\frac{1}{z-\zeta} + \sum_{l=0}^{m-1} \frac{z^l}{\zeta^{l+1}} = \frac{z^m}{\zeta^m (z-\zeta)}$, for all $\zeta \neq z, 0$. $\Longrightarrow$

$$\leq \frac{1}{\pi^2} \int_{D_r} \left( \int_{\mathbb{C}} \frac{1}{|z-\zeta|} \frac{|\bar{\partial} \left( \psi_k(\zeta)\eta(\zeta)u(\zeta) \right)|}{|\zeta|^m} dv_\zeta \right)^2 V(z) dv_z$$

$$\leq \frac{1}{\pi^2} \left\| I_1 \left( \frac{|\bar{\partial} \left( \psi_k \eta u \right)|}{|\cdot|^m} \right) \right\|^2_{L^2_V(\mathbb{C})}$$

$$\leq \frac{C}{\pi^2} \|V\|^2_{L^2(D_r)} \int_{\mathbb{C}} \frac{|\bar{\partial} \left( \psi_k(z)\eta(z)u(z) \right)|^2}{|z|^{2m}} \frac{dv_z}{V(z)}.$$

# Proof of the main theorem, continued

Let $r << 1$ such that $\|V\|^2_{L^2(D_r)} \leq \frac{\pi^2}{2C}$ and $V \geq C_r$ on $D_{2r}$.

$$\begin{aligned}
&\int_{D_r} \frac{|\psi_k(z)|^2|u(z)|^2}{|z|^{2m}} V(z) dv_z \\
\leq &\frac{1}{2}\int_{\mathbb{C}} \frac{|\bar{\partial}\psi_k(z)|^2|u(z)|^2}{|z|^{2m}V(z)} dv_z + \frac{1}{2}\int_{D_r} \frac{|\psi_k(z)|^2|\bar{\partial}u(z)|^2}{|z|^{2m}V(z)} dv_z \\
&+ \frac{1}{2}\int_{D_{2r}\setminus D_r} \frac{|\bar{\partial}\left(\eta(z)u(z)\right)|^2}{|z|^{2m}V(z)} dv_z
\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}
&\int_{D_r} \frac{|\psi_k(z)|^2|u(z)|^2}{|z|^{2m}} V(z) dv_z \\
\leq &\int_{\mathbb{C}} \frac{|\bar{\partial}\psi_k(z)|^2|u(z)|^2}{|z|^{2m}V(z)} dv_z + \int_{D_{2r}\setminus D_r} \frac{|\bar{\partial}\left(\eta(z)u(z)\right)|^2}{|z|^{2m}V(z)} dv_z.
\end{aligned}$$

# Proof of the main theorem, continued

Note that

$$\lim_{k\to\infty}\int_{\mathbb{C}}\frac{|\bar\partial\psi_k(z)|^2|u(z)|^2}{|z|^{2m}V(z)}dv_z=0.$$

Indeed,

- $\bar\partial\psi_k$ is supported on $D_{\frac{2}{k}}\setminus D_{\frac{1}{k}}$, and $|\nabla\psi_k|\approx k$;
- $V>C_r$ on $D_{\frac{2}{k}}(\subset D_r)$.

$\Longrightarrow$

$$\int_{\mathbb{C}}\frac{|\bar\partial\psi_k(z)|^2|u(z)|^2}{|z|^{2m}V(z)}dv_z\le\int_{\frac{1}{k}<|z|<\frac{2}{k}}\frac{k^2|u(z)|^2}{|z|^{2m}C_r}dv_z$$
$$\le\frac{k^{2m+2}}{C_r}\int_{|z|<\frac{2}{k}}|u(z)|^2dv_z\to 0$$

as $k\to\infty$.

# Proof of the main theorem, continued

Let $k \to \infty$. Then for each $m \in \mathbb{Z}^+$,

$$\int_{D_r} \frac{|u(z)|^2}{|z|^{2m}} V(z) dv_z \leq \int_{D_{2r} \setminus D_r} \frac{|\bar{\partial}(\eta(z)u(z))|^2}{|z|^{2m} V(z)} dv_z.$$

$\implies$

$$\frac{1}{|\frac{r}{2}|^{2m}} \int_{D_{\frac{r}{2}}} |u(z)|^2 V(z) dv_z \leq \frac{1}{r^{2m}} \int_{D_{2r} \setminus D_r} \frac{|\bar{\partial}(\eta(z)u(z))|^2}{V(z)} dv_z.$$

$\implies$

$$2^{2m} \int_{D_{\frac{r}{2}}} |u(z)|^2 V(z) dv_z \leq \tilde{C}_r \|u\|^2_{H^1(D_{2r})},$$

Letting $m \to \infty$, we have $u = 0$ on $D_{\frac{r}{2}}$.

# Further remarks when $V = \frac{C}{|z|}$

Note that $\frac{1}{|z|} \notin L^2_{loc}(\mathbb{C})$.

**Remarks:**

- The strong UCP holds for $|\bar{\partial} u| \leq V|u|$ if $n = N = 1$ and $V = \frac{C}{|z|}$ (Pan, Comm PDE. 1992).

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**Remarks:**

- The strong UCP holds for $|\bar{\partial} u| \leq V|u|$ if $n = N = 1$ and $V = \frac{C}{|z|}$ (Pan, Comm PDE. 1992).
- The strong UCP fails for $V = \frac{C}{|z|}$ when $N = 2$ and $C$ is large. (Alinhac-Baouendi, Duke Math J. 1990) and (Pan-Wolff, J. Geom Anal 1998).

Thanks!